Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 27/9/14.
Duration: 2.30 p.m.–4.00 p.m.
Maximum Marks: 100

1. Let $\{\boldsymbol{p}_i\}$ and $\{\boldsymbol{q}_i\}$, $i = 1, 2, 3$, be sets of vectors in $\Re^3$. Determine if the set $\{\boldsymbol{p}_i \otimes \boldsymbol{q}_i\}$, $i = 1, 2, 3$, is linearly dependent or independent if (25)
   (a) $\{\boldsymbol{p}_i\}$ is linearly independent, and $\{\boldsymbol{q}_i\}$ is linearly independent;
   (b) $\{\boldsymbol{p}_i\}$ is linearly independent, but $\{\boldsymbol{q}_i\}$ is not;
   (c) $\{\boldsymbol{q}_i\}$ is linearly independent, but $\{\boldsymbol{p}_i\}$ is not;
   (d) $\{\boldsymbol{p}_i\}$ is linearly dependent, and $\{\boldsymbol{q}_i\}$ is also linearly dependent.

2. Let $\boldsymbol{W} \in \text{Skw}$ and let $\boldsymbol{w}$ be its axial vector. Find the polar decomposition of $\boldsymbol{I} + \boldsymbol{W}$ *explicitly*, i.e., the factors $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ should be explicit functions of $\boldsymbol{w}$ and/or $\boldsymbol{W}$, which one can compute using a calculator if $\boldsymbol{W}$ is given in numerical form. (25)

3. Let the underlying vector space dimension be 2, and let $\boldsymbol{W} \in \text{Skw}$ in this case, i.e., (25)
$$\boldsymbol{W} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}.$$
   (a) Given that the first and last invariants are the trace and the determinant, find an explicit expression for $e^{\boldsymbol{W}}$, which one can compute using a calculator if α is given.
   (b) Using this explicit expression, find $De^{\boldsymbol{W}}(\boldsymbol{W})[\boldsymbol{U}]$.

4. Superposed dots denote material time derivatives, and $(\boldsymbol{F}, \boldsymbol{L})$ denote the deformation and velocity gradients. (25)
   (a) Derive a relation between $\dot{\boldsymbol{F}}$ and $\boldsymbol{F}$.
   (b) Derive a relation between $\ddot{\boldsymbol{F}}$ and $\boldsymbol{\nabla}_x \boldsymbol{a}$, where $\boldsymbol{a}$ is the acceleration.
   (c) Using the above relations or independently, find the material derivative of $\boldsymbol{\nabla}_x \cdot \boldsymbol{v}$ in terms of $\boldsymbol{a}$, and possibly $\boldsymbol{L}$ and $\boldsymbol{F}$.

Some relevant formulae

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}), \quad (\boldsymbol{w}/|\boldsymbol{w}|\,, \boldsymbol{q}, \boldsymbol{r} \text{ orthonormal}),$$
$$W_{ij} = -\epsilon_{ijk} w_k, \quad w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk},$$
$$e^{\boldsymbol{T}} = \boldsymbol{I} + \boldsymbol{T} + \frac{1}{2!}\boldsymbol{T}^2 + \cdots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$