Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 6/10/18.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

1. Given a *fixed* tensor $\boldsymbol{W}_0 \in \mathrm{Skw}$ with $\boldsymbol{w}_0$ as its axial vector, find the most general form of $\boldsymbol{S} \in \mathrm{Sym}$ and $\boldsymbol{W} \in \mathrm{Skw}$ (that could depend on $\boldsymbol{W}_0$ and/or $\boldsymbol{w}_0$) that satisfy (35)

$$
\begin{aligned}
(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{W}_0\boldsymbol{u}) &= 0 \quad \forall \boldsymbol{u} \in V, \\
(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{W}_0\boldsymbol{u}) &= 0 \quad \forall \boldsymbol{u} \in V.
\end{aligned}
$$

You may assume that if $\hat{\boldsymbol{S}} \in \mathrm{Sym}$, then $(\boldsymbol{u}, \hat{\boldsymbol{S}}\boldsymbol{u}) = 0$ for all $\boldsymbol{u}$ if and only if $\hat{\boldsymbol{S}} = \boldsymbol{0}$. Deduce any other result that you require.

2. Let $\boldsymbol{W} \in \mathrm{Skw}$. Find $De^{\boldsymbol{W}^2}(\boldsymbol{W})[\boldsymbol{U}]$. Your answer should not be in the form of an infinite series. (35)

3. The velocity distribution for a point vortex in which each fluid particle moves in a circle around the origin is given by (30)

$$
\begin{aligned}
u_x &= -\frac{\Gamma y}{2\pi(x^2+y^2)}, \\
u_y &= \frac{\Gamma x}{2\pi(x^2+y^2)},
\end{aligned}
$$

where Γ is a constant. We want to find the equation of motion corresponding to the above velocity field. Assuming the motion to be given by

$$
\begin{aligned}
x &= f_1(X)\cos\frac{\Gamma t f_4(X^2+Y^2)}{2\pi} - f_2(Y)\sin\frac{\Gamma t f_4(X^2+Y^2)}{2\pi}, \\
y &= f_2(X)\sin\frac{\Gamma t f_4(X^2+Y^2)}{2\pi} + f_3(Y)\cos\frac{\Gamma t f_4(X^2+Y^2)}{2\pi},
\end{aligned}
$$

determine the functions f_1, f_2, f_3 and f_4. Guesswork for the functions is allowed as long as you show that all the governing equations are satisfied.

Some relevant formulae

$$
\begin{aligned}
\boldsymbol{W} &= |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}), \quad (\boldsymbol{w}/|\boldsymbol{w}|, \boldsymbol{q}, \boldsymbol{r} \text{ orthonormal}), \\
W_{ij} &= -\epsilon_{ijk} w_k, \\
w_i &= -\frac{1}{2}\epsilon_{ijk} W_{jk}, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \\
\frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots.
\end{aligned}
$$