Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 2/10/19.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

1. Given two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ having the same magnitude, find a tensor transformation $\boldsymbol{T}$ that transforms $\boldsymbol{u}$ to $\boldsymbol{v}$, i.e., $\boldsymbol{v} = \boldsymbol{T}\boldsymbol{u}$ (obviously $\boldsymbol{T}$ is a function of $\boldsymbol{u}$, $\boldsymbol{v}$, and any other vectors that you may wish to introduce), and such that (30)

   (a) $\det \boldsymbol{T} = 0$.

   (b) $\det \boldsymbol{T} = 1$.

   (c) $\boldsymbol{T} \in \mathrm{Sym}$ (Hint: Try spectral resolution).

   Show that the tensor $\boldsymbol{T}$ that you construct in each case satisfies the given properties. This problem shows that a tensor transformation that maps a given vector $\boldsymbol{u}$ to another given vector $\boldsymbol{v}$ is not unique.

2. The disc shown in Fig. 1 rotates with a constant angular speed ω about $\boldsymbol{e}_3$, and simultaneously 'pulsates', i.e., each point on a circle of radius R moves radially with respect to a coordinate system fixed to the rotating disc so as to lie on a circle of radius $r = R[1 + \epsilon(t)]$, where $\epsilon(0) = 0$. The angular speed ω is a constant. Find the velocities $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$, accelerations $\boldsymbol{a}$, $\tilde{\boldsymbol{a}}$, deformation gradient $\boldsymbol{F}$ along with its polar decomposition factors $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$, and the strain tensor $\boldsymbol{E}(\boldsymbol{X}, t)$, all with respect to the fixed $\boldsymbol{e}_1$–$\boldsymbol{e}_2$ coordinate system; assume the problem to be two-dimensional so that matrices are 2×2 etc. You may write your answers in terms of matrices which you need not multiply, but whose individual expressions should be given. (35)

3. Let $\boldsymbol{W} \in \mathrm{Skw}$, and let $\boldsymbol{w}$ be its axial vector. Find a relation for $\mathbf{cof}\,\boldsymbol{W}$ in terms of $\boldsymbol{w}$. Next find $e^{\mathbf{cof}\,\boldsymbol{W}}$ in terms of $\boldsymbol{w}$; this expression should have a finite number of terms (Hint: See if you can express it as $\boldsymbol{I} + \alpha \boldsymbol{w} \otimes \boldsymbol{w}$, where α is a coefficient you have to determine). Lastly, find $De^{\mathbf{cof}\,\boldsymbol{W}}(\boldsymbol{w})[\boldsymbol{u}]$ where $\boldsymbol{u}$ is a perturbation on $\boldsymbol{w}$. (35)

Some relevant formulae

$$\det \boldsymbol{T} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{ip}T_{jq}T_{kr},$$

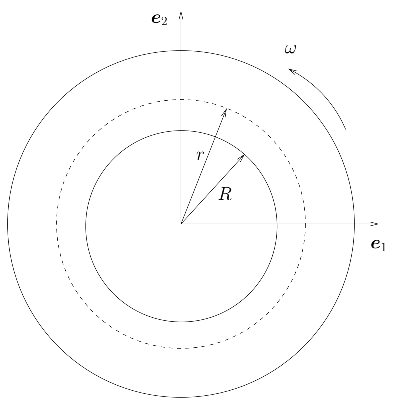

Figure 1: Pulsating spinning disc.

$$
\begin{gathered}
(\mathbf{cof}\,\boldsymbol{T})^T = I_2\boldsymbol{I} - (\operatorname{tr}\boldsymbol{T})\boldsymbol{T} + \boldsymbol{T}^2, \\
W_{ij} = -\epsilon_{ijk} w_k, \\
w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk}, \\
\boldsymbol{R}(\boldsymbol{w},\alpha) = \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|}\sin\alpha\,\boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1-\cos\alpha)\boldsymbol{W}^2, \\
\boldsymbol{R} = \bar{\boldsymbol{e}}_k \otimes \boldsymbol{e}_k^*, \\
(\mathbf{cof}\,\boldsymbol{T})_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jpq} T_{mp} T_{nq}, \\
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \\
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \\
\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots.
\end{gathered}
$$