Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 30/9/00.
Duration: 3.00 p.m.–4.30 p.m.
Maximum Marks: 100

1. Show that if $\boldsymbol{B}$ is an invertible tensor, then the the eigenvalues of $\boldsymbol{BAB}^{-1}$ and $\boldsymbol{A}$ are identical. If $\boldsymbol{n}$ is an eigenvector of $\boldsymbol{A}$, then what is the corresponding eigenvector (i.e., eigenvector corresponding to the same eigenvalue) of $\boldsymbol{BAB}^{-1}$? (10)

2. We are interested in finding the eigenvalues of $\boldsymbol{W} \in \text{Skw}$ in terms of its axial vector $\boldsymbol{w}$. Toward this end (15)

   (a) Using the representation $\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r})$, find $\operatorname{tr}(\boldsymbol{W}^2)$.

   (b) Find the principal invariants, $I_1 = \operatorname{tr}\boldsymbol{W}$, $I_2 = 0.5[(\operatorname{tr}\boldsymbol{W})^2 - \operatorname{tr}(\boldsymbol{W}^2)]$, and $I_3 = \det\boldsymbol{W}$.

   (c) Use the characteristic equation to find the eigenvalues.

3. The Fibonacci numbers satisfy the recurrence relation $F_{n+1} = F_n + F_{n-1}$ with $F_1 = F_2 = 1$. In matrix form this recurrence relation can be written as (30)

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

   where $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Use the above matrix form to first express $\left[\begin{smallmatrix} F_{n+1} \\ F_n \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} F_2 \\ F_1 \end{smallmatrix}\right]$, and then use this to find an explicit expression for F_n in terms of the eigenvalues λ_1 and λ_2 of A.

4. Find $\partial\phi(\boldsymbol{T})/\partial\boldsymbol{T}$, where $\phi(\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{T}^{-1}\boldsymbol{T}^{-1})$. (20)

5. Given a vector field $\boldsymbol{v} : V \to \Re^3$ over the deformed configuration V, its Piola transform is the vector field $\boldsymbol{u} : V_0 \to \Re^3$ defined over the reference configuration by the relation (25)

$$\boldsymbol{u}(\boldsymbol{X}) = J\boldsymbol{F}^{-1}(\boldsymbol{X})\boldsymbol{v}(\boldsymbol{x}).$$

   We wish to find a relation between the divergences $\boldsymbol{\nabla}_X \cdot \boldsymbol{u}$ and $\boldsymbol{\nabla}_x \cdot \boldsymbol{v}$. With a view towards this

   (a) If $\boldsymbol{T}$ is a SOT and $\boldsymbol{a}$ a vector, find a relation for $\boldsymbol{\nabla} \cdot (\boldsymbol{T}^t\boldsymbol{a})$ in terms of the divergence of $\boldsymbol{T}$ and the gradient of $\boldsymbol{a}$.

   (b) Using this relation and the fact that $\boldsymbol{\nabla}_X \cdot \mathbf{cof}\,\boldsymbol{F} = \boldsymbol{0}$, show that the divergences of the two vector fields are related by

$$\boldsymbol{\nabla}_X \cdot \boldsymbol{u}(\boldsymbol{X}) = J\boldsymbol{\nabla}_x \cdot \boldsymbol{v}(\boldsymbol{x}).$$

   Note that this relation is similar to the relation between the divergences of tensor fields.