Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 7/10/22.
Duration: 9.30 a.m.–11.30 noon
Maximum Marks: 100

1. Let the underlying vector space be two-dimensional. Let $\{\boldsymbol{u}, \boldsymbol{v}\}$ be a *given* set of orthonormal vectors in this two-dimensional plane (say the x-y plane). It is clear that $\boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{v} \otimes \boldsymbol{v} \in$ Sym. Is the converse true? That is, can every $\boldsymbol{S} \in$ Sym be represented as $\alpha \boldsymbol{u} \otimes \boldsymbol{u} + \beta \boldsymbol{v} \otimes \boldsymbol{v}$? If not, then determine an appropriate basis for Sym in terms of $(\boldsymbol{u}, \boldsymbol{v})$, and show that every $\boldsymbol{S} \in$ Sym can be written in terms of your proposed basis. (Hint: Dimensionality of the spaces involved). (25)

2. Let $\boldsymbol{W} \in$ Skw, and let $\boldsymbol{w}$ be its axial vector. (35)

   (a) Using the relation $(\mathbf{cof}\,\boldsymbol{T})(\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u}) \times (\boldsymbol{T}\boldsymbol{v})$ (and using no other method), find an expression for $\mathbf{cof}\,\boldsymbol{W}$ in terms of $\boldsymbol{w}$; justify all your steps.

   (b) Let $\boldsymbol{H} := \mathbf{cof}\,(\boldsymbol{W} - \alpha \boldsymbol{W}^3)$, where $\alpha > 0$, and $\boldsymbol{W}$ is a skew-symmetric tensor whose axial vector $\boldsymbol{w}$ is of unit magnitude. Find $\boldsymbol{H}$ in terms of the axial vector of $\boldsymbol{w}$ and α. Find the factors $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ in the polar decomposition of $-(\boldsymbol{I} + \boldsymbol{H})^{-1}$ in terms of $\boldsymbol{w}$ and α without 'guesswork'.

3. If $\boldsymbol{\lambda} = \boldsymbol{dx}/ds$ represents the unit tangent to a curve, find its material derivative $D\boldsymbol{\lambda}/Dt$ in terms of $\boldsymbol{\lambda}$ and the velocity gradient $\boldsymbol{L}$. You may directly use the relation $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$, but need to derive any other relation that you need. Using the expression that you have derived, deduce necessary and sufficient conditions on $D\boldsymbol{\lambda}/Dt$ in terms of the angular velocity vector $\boldsymbol{w}(t)$ for the motion to be rigid (you may directly use the results for rigid motion that we have derived). (40)

Some relevant formulae

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}), \quad (\boldsymbol{w}/|\boldsymbol{w}|\,, \boldsymbol{q}, \boldsymbol{r} \text{ orthonormal}),$$

$$W_{ij} = -\epsilon_{ijk} w_k,$$

$$w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk},$$

$$(\mathbf{cof}\,\boldsymbol{T})_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jpq} T_{mp} T_{nq},$$