Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 5/10/24.
Duration: 10.00 a.m.–12.00 noon.
Maximum Marks: 100

1. Let $\boldsymbol{w}$ be the axial vector of $\boldsymbol{W} \in \text{Skw}$, and let $\boldsymbol{u}, \boldsymbol{v} \in V$. (35)

   (a) Using indicial notation or otherwise, find an expression for $\textbf{cof}\,\boldsymbol{W}$ in terms of $\boldsymbol{w}$.

   (b) Using the above result or otherwise, find a necessary and sufficient condition for the linear independence of $\{\boldsymbol{W}\boldsymbol{u}, \boldsymbol{W}\boldsymbol{v}, \boldsymbol{w}\}$ in terms of the linear independence of $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

2. Let $\boldsymbol{n}$ represent a unit normal to a plane. (35)

   (a) Find an expression for the tensor $\boldsymbol{P}$ that projects a vector $\boldsymbol{u}$ onto the plane with unit normal $\boldsymbol{n}$ (thus, $\boldsymbol{P}\boldsymbol{u}$ is a vector that lies in the plane with unit normal $\boldsymbol{n}$) in terms of $\boldsymbol{n} \otimes \boldsymbol{n}$.

   (b) Find an explicit expression for $e^{\boldsymbol{P}}$ in terms of $\boldsymbol{P}$ which has a finite number of terms (Hint: $e^{\alpha} = \sum_{n=0}^{\infty} \alpha^n/n!$)

   (c) Using the above results, find $De^{\boldsymbol{P}}(\boldsymbol{P})[\boldsymbol{U}]$.

3. Let $\boldsymbol{v}$ denote the velocity vector, and let $\boldsymbol{D}$ and $\boldsymbol{W}$ denote the rate of deformation and vorticity tensors, respectively. Let V denote a closed volume, and S its surface. (30)

   (a) Using indicial notation, find a relation between $\boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{v} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{v}]$, $\boldsymbol{\nabla}\boldsymbol{v} : (\boldsymbol{\nabla}\boldsymbol{v})^T$, and a term that is a function of $\boldsymbol{\nabla} \cdot \boldsymbol{v}$.

   (b) Using the expression you obtained in part (a) above, find an expression for $\int_V \boldsymbol{D} : \boldsymbol{D}\, dV$ in terms of $\int_V \boldsymbol{W} : \boldsymbol{W}\, dV$ and other terms (Hint: Use the relation between $\boldsymbol{D}$, $\boldsymbol{W}$, and the velocity gradient).

   (c) One of the volume integrals in the expression you obtained in part (b) above can be converted to a surface integral using the divergence theorem. Carry out this conversion.

   (d) If $\boldsymbol{\omega}$ denotes the vorticity vector, in one of the homeworks, we saw that $\boldsymbol{\omega}$ is twice the axial vector of $\boldsymbol{W}$. Use this information to find a relation between $\boldsymbol{W} : \boldsymbol{W}$ and $\boldsymbol{\omega} \cdot \boldsymbol{\omega}$.

Some relevant formulae

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}), \quad (\boldsymbol{w}/|\boldsymbol{w}|\,, \boldsymbol{q}, \boldsymbol{r} \text{ orthonormal}),$$
$$W_{ij} = -\epsilon_{ijk} w_k,$$
$$w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk},$$
$$(\textbf{cof}\,\boldsymbol{T})_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jpq} T_{mp} T_{nq},$$