Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 27/9/25.
Duration: 9.00 a.m.–11.30 a.m.
Maximum Marks: 100

1. Let $\boldsymbol{S} \in \text{Sym}$, and let $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\}$ be an orthonormal basis. (30)

   (a) If
   $$(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = 0 \quad \forall \boldsymbol{u} \in V,$$
   then find the most general form of $\boldsymbol{S}$.

   (b) Let $\boldsymbol{T}$ be given by
   $$\boldsymbol{T} = \beta\left[\boldsymbol{p} \otimes \boldsymbol{p} + \cos\alpha(\boldsymbol{q} \otimes \boldsymbol{q} + \boldsymbol{r} \otimes \boldsymbol{r}) + \sin\alpha(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r})\right],$$
   and let $\boldsymbol{u}$ be an arbitrary vector. Find a relation between $(\boldsymbol{T}\boldsymbol{u}, \boldsymbol{T}\boldsymbol{u})$ and $(\boldsymbol{u}, \boldsymbol{u})$ (Hint: $(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{u}, \boldsymbol{I}\boldsymbol{u})$ where $\boldsymbol{I}$ is expressed in an appropriate basis).

   (c) Using the results of Items (a) and (b) above, find the values of β for which $\boldsymbol{T}$ is orthogonal. *Justify* your reasoning while deducing that $\boldsymbol{T} \in \text{Orth}$, using say the results of part (a).

   (d) Let $\alpha = \pi$ and $\beta = 1$. Determine the eigenvalues/eigenvectors of $\boldsymbol{T}$ (one choice of orthonormal eigenvectors if they are not unique), and determine whether it is positive-definite for this choice of parameters.

2. Let $\boldsymbol{W} \in \text{Skw}$. Find an expression for $e^{\boldsymbol{W}^2}$ as a function of $\boldsymbol{W}$ and $|\boldsymbol{w}|^2$ that has a finite number of terms. Next, using $\boldsymbol{W} : \boldsymbol{W} = \alpha \boldsymbol{w} \cdot \boldsymbol{w}$, where you have to determine α, find $De^{\boldsymbol{W}^2}(\boldsymbol{W})[\boldsymbol{U}]$. (40)

3. Let the displacement field with respect to a cylindrical coordinate system be given by $(u_r, u_\theta, u_z) = (0, \alpha(t)RZ, 0)$, where t denotes time, and R and Z refer to the radial and axial coordinates of a point in the undeformed configuration. (30)

   (a) Using appropriate tensorial transformations, transform the displacement vector to Cartesian components (u_x, u_y, u_z).

   (b) Find the mapping $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, and the associated deformation gradient $\boldsymbol{F}(\boldsymbol{X}, t)$ and $J(\boldsymbol{x}, t)$. Is the deformation isochoric?

   (c) State the Lagrangian strain $\boldsymbol{E}(\boldsymbol{X}, t)$ (you need not multiply matrices; just state in symbolic form), and the small strain tensor $\boldsymbol{\epsilon}$ (state this in explicit form).

   (d) Find the Lagrangian and Eulerian velocity functions $\tilde{\boldsymbol{v}}(\boldsymbol{X}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$ (Hint: Use the fact that $z = Z$ while carrying out the inversion of the mapping).

   (e) Using the Eulerian form, find $\boldsymbol{\nabla}_x \cdot \boldsymbol{v}$ (Do not find the entire velocity gradient tensor), and verify your earlier conclusion about isochoricity.

Some relevant formulae

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}), \quad (\boldsymbol{w}/|\boldsymbol{w}|, \boldsymbol{q}, \boldsymbol{r} \text{ orthonormal}),$$
$$W_{ij} = -\epsilon_{ijk} w_k,$$
$$w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk},$$
$$(\mathbf{cof}\,\boldsymbol{T})_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jpq} T_{mp} T_{nq},$$