Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 6/10/01.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

1. Which of the spaces Sym, Psym, Skw, Orth$^+$ are linear subspaces of Lin? Justify. Evaluate if the matrices (20)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

constitute a basis for Sym. If not, then give a 'canonical' basis for Sym.

2. Using the representation $\boldsymbol{W} = |\boldsymbol{w}| (\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r})$, where $\{\boldsymbol{w}/|\boldsymbol{w}|, \boldsymbol{q}, \boldsymbol{r}\}$ form an orthonormal basis, find the axial vector of $\boldsymbol{Q}^t \boldsymbol{W} \boldsymbol{Q}$, where $\boldsymbol{Q}$ is a proper orthogonal tensor. (20)

3. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the usual canonical basis for $\Re^3$, and $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ be a set of vectors. (25)

   (a) Show that
   $$\boldsymbol{e}_1 \otimes \boldsymbol{a} + \boldsymbol{e}_2 \otimes \boldsymbol{b} + \boldsymbol{e}_3 \otimes \boldsymbol{c} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

   (b) Using the above result, show that
   $$\det\left[\boldsymbol{a} \otimes \boldsymbol{x} + \boldsymbol{b} \otimes \boldsymbol{y} + \boldsymbol{c} \otimes \boldsymbol{z}\right] = \left[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\right]\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\right].$$

   (c) If $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}_{1'}, \boldsymbol{e}_{2'}, \boldsymbol{e}_{3'}\}$ are two sets of orthonormal bases, show that
   $$\boldsymbol{Q} = \boldsymbol{e}_{1'} \otimes \boldsymbol{e}_1 + \boldsymbol{e}_{2'} \otimes \boldsymbol{e}_2 + \boldsymbol{e}_{3'} \otimes \boldsymbol{e}_3,$$
   is a proper orthogonal tensor. Also show that $\boldsymbol{e}_{i'} = \boldsymbol{Q}\boldsymbol{e}_i$.

4. If $\boldsymbol{x}$ is the position vector of a point, and $\boldsymbol{t}$ is the axial vector of $(\boldsymbol{T} - \boldsymbol{T}^t)$, show that (15)
$$\int_V \left[\boldsymbol{x} \times (\boldsymbol{\nabla} \cdot \boldsymbol{T}) + \boldsymbol{t}\right] dV = \int_S \boldsymbol{x} \times (\boldsymbol{T}\boldsymbol{n})\, dS.$$
(Hint: $w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk}$.)

5. Starting from the relation (20)
$$J = \epsilon_{ijk} \frac{\partial \chi_1}{\partial X_i} \frac{\partial \chi_2}{\partial X_j} \frac{\partial \chi_3}{\partial X_k}.$$
show that $DJ/Dt = J(\boldsymbol{\nabla} \cdot \boldsymbol{v})$. (Hint: Use the relation $\epsilon_{pqr}(\det \boldsymbol{T}) = \epsilon_{ijk} T_{pi} T_{qj} T_{rk}$.)