Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 5/10/02.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

1. Prove that if $\boldsymbol{T} : \boldsymbol{S} = 0$ for every symmetric tensor $\boldsymbol{S}$, then $\boldsymbol{T} \in \text{Skw}$. (15)

2. Deriving any results that you might need on the way, show that (25)

$$\boldsymbol{R}(\boldsymbol{w}, \alpha) = \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|} \sin\alpha\, \boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1 - \cos\alpha)\boldsymbol{W}^2,$$

where $\boldsymbol{W}$ is the skew-symmetric tensor with $\boldsymbol{w}$ as the axis, rotates any vector in the plane perpendicular to $\boldsymbol{w}$ through an angle α.

3. Our goal in this problem is to find explicit expressions for $\boldsymbol{R}$ and $\boldsymbol{U}$ in the polar decompostion $\boldsymbol{S} = \boldsymbol{R}\boldsymbol{U}$, where $\boldsymbol{S}$ is a *nonsingular* symmetric tensor, in terms of powers of $\boldsymbol{S}$. (35)

   (a) Let V be an n-dimensional vector space with $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ as a basis, and let $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ be given by

$$\boldsymbol{f}_i = \beta_{ij}\boldsymbol{e}_j, \quad \beta_{ij} \in \Re,\ i, j = 1, n,$$

   where β_{ij} are such that $\det[\beta_{ij}] \neq 0$. We have shown in class that under such conditions, $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ is a basis. Hence, we can express the basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ in terms of the basis vectors $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ as

$$\boldsymbol{e}_i = \gamma_{ij}\boldsymbol{f}_j, \quad \gamma_{ij} \in \Re,\ i, j = 1, n.$$

   Show that $\gamma_{ij}\beta_{jm} = \delta_{im}$, i.e., the matrix of coefficients $[\gamma_{ij}]$ is the inverse of the matrix of coefficients $[\beta_{ij}]$.

   (b) Starting with the spectral resolution of $\boldsymbol{S} \in \text{Sym}$, namely,

$$\boldsymbol{S} = \lambda_1 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*,$$

   find the spectral resolutions of $\boldsymbol{R}$ and $\boldsymbol{U}$ (Hint: Look for tensors of the form

$$|\lambda_1|\, \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + |\lambda_2|\, \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + |\lambda_3|\, \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*.)$$

   (c) Consider the case when the two eigenvalues are repeated, say, $\lambda_2 = \lambda_3$. Express $\{\boldsymbol{I}, \boldsymbol{S}\}$ in terms of $\{\boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^*, (\boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*)\}$, in the form $\boldsymbol{f}_i = \beta_{ij}\boldsymbol{e}_j$, and show that $\det[\beta_{ij}] \neq 0$. Now using the result of (a), express $\{\boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^*, (\boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*)\}$ in terms of $\{\boldsymbol{I}, \boldsymbol{S}\}$.

   (d) Substitute these expressions in the polar resolutions of $\boldsymbol{R}$ and $\boldsymbol{U}$ (for the case $\lambda_2 = \lambda_3$), and find explicit expressions for $\boldsymbol{R}$ and $\boldsymbol{U}$ as a polynomial expression involving powers of $\boldsymbol{S}$.

4. Using the relations $DJ/Dt = J(\boldsymbol{\nabla} \cdot \boldsymbol{v})$, $D\boldsymbol{F}/Dt = \boldsymbol{L}\boldsymbol{F}$, and $\boldsymbol{C} = \boldsymbol{F}^t\boldsymbol{F}$, show that (25)

$$\frac{DJ}{Dt} = \frac{1}{2} J\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}}.$$