Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 23/9/03.
Duration: 3.30 p.m.–5.00 p.m.
Maximum Marks: 100

1. Show that $\boldsymbol{Q} \in \text{Orth}^+ \cap \text{Sym} - \{\boldsymbol{I}\}$ if and only if it is of the form $2\boldsymbol{e} \otimes \boldsymbol{e} - \boldsymbol{I}$ where $\boldsymbol{e}$ is a unit vector. (25)

2. A tensor $\boldsymbol{P}$ is a *projection* if it satisfies the relation $\boldsymbol{P}^2 = \boldsymbol{P}$. (35)

   (a) Show that if $\boldsymbol{P}$ is a projection then $\boldsymbol{I} - \boldsymbol{P}$ is also a projection.

   (b) Find if $\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ and $\boldsymbol{e}_1 \otimes (\boldsymbol{e}_1 + \boldsymbol{e}_2)$ are projections.

   (c) To derive a characterization for projections (similar to the spectral resolution for symmetric tensors), recall that any tensor $\boldsymbol{T}$ can be expressed as (in 2D)

   $$\boldsymbol{T} = \lambda_1 \boldsymbol{n}_1 \otimes \boldsymbol{N}_1 + \lambda_2 \boldsymbol{n}_2 \otimes \boldsymbol{N}_2,$$

   where λ_i^2 are the eigenvalues of $\boldsymbol{T}^t\boldsymbol{T}$, and $\boldsymbol{n}_i$ and $\boldsymbol{N}_i$ are orthonormal sets of vectors. Also recall that the λ_is are nonnegative. Using these properties, find conditions on the λ_i and $(\boldsymbol{n}_i \cdot \boldsymbol{N}_j)$, $i, j = 1, 2$, in order for $\boldsymbol{T}$ to be a projection. Substituting these conditions into the expression for $\boldsymbol{T}$, find the required characterization.

   (d) Based on the deduced characterization, is a projection always symmetric? If not, under what conditions is it symmetric?

   (e) Using your characterization, find an orthogonal $\boldsymbol{Q}$ that diagonalises $\boldsymbol{P} \in \text{Sym}$, i.e., $\boldsymbol{QPQ}^t$ should be diagonal.

3. Find $\partial\phi/\partial\boldsymbol{T}$, where $\phi = (\det \boldsymbol{T})\boldsymbol{T}^{-1} : \boldsymbol{T}^{-1}$. (25)

4. Show that the infinitesimal strain $\boldsymbol{\epsilon} = 0.5[\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u})^t]$ is zero at all $\boldsymbol{X} \in V_0$ if and only if $\boldsymbol{u}$ is of the form $\boldsymbol{u} = \boldsymbol{WX} + \boldsymbol{c}$ where $\boldsymbol{W}$ is a skew tensor. You may assume $\boldsymbol{u}$ to be twice differentiable. (Note that if $\boldsymbol{w}$ is the axial vector of $\boldsymbol{W}$, the above relation can also be written as $\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{X} + \boldsymbol{c}$.) (15)