Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 22/9/05.
Duration: 3.30 p.m.–5.00 p.m.
Maximum Marks: 100

1. Do the following: (40)

   (a) Show that for $\boldsymbol{R} \in \text{Orth}^+$, $\operatorname{tr} \boldsymbol{R}$ is the only "independent" invariant among the three principal invariants.

   (b) Substitute $\boldsymbol{W} = |\boldsymbol{w}| (\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r})$, $\quad (\boldsymbol{w}/|\boldsymbol{w}|, \boldsymbol{q}, \boldsymbol{r}$ orthonormal) into the equation

   $$\boldsymbol{R}(\boldsymbol{w}, \alpha) = \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|} \sin\alpha\, \boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1 - \cos\alpha)\boldsymbol{W}^2,$$

   and obtain an equation for $\boldsymbol{R} \in \text{Orth}^+$ in terms of α, $\boldsymbol{I}$, $\boldsymbol{e} \otimes \boldsymbol{e}$ (where $\boldsymbol{e} := \boldsymbol{w}/|\boldsymbol{w}|$), $\boldsymbol{r} \otimes \boldsymbol{q}$ and $\boldsymbol{q} \otimes \boldsymbol{r}$. Find $\operatorname{tr} \boldsymbol{R}$ using this equation.

   (c) Prove the following: Two proper orthogonal tensors $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ have the same trace, i.e., $\operatorname{tr} \boldsymbol{Q}_1 = \operatorname{tr} \boldsymbol{Q}_2$, if and only if there exists an orthogonal tensor $\boldsymbol{Q}_0$ (which means that either $\boldsymbol{Q}_0$ or $-\boldsymbol{Q}_0$ is a rotation) such that $\boldsymbol{Q}_2 = \boldsymbol{Q}_0 \boldsymbol{Q}_1 \boldsymbol{Q}_0^t$. (Hint: To prove the "only if" part, use the representation for a proper orthogonal tensor that you derive in part (b) above to find an explicit formula for $\boldsymbol{Q}_0$.)

   (d) Using the representation that you derive in part (b) above, prove that $\boldsymbol{R} \in \text{Orth}^+ \cap \text{Sym} - \{\boldsymbol{I}\}$ if and only if it is of the form $2\boldsymbol{e} \otimes \boldsymbol{e} - \boldsymbol{I}$.

2. Let $\boldsymbol{F}$ be the deformation gradient. Using the expressions (30)

   $$\mathbf{cof}\,\boldsymbol{F} = \begin{bmatrix} F_{22}F_{33} - F_{23}F_{32} & F_{23}F_{31} - F_{21}F_{33} & F_{21}F_{32} - F_{22}F_{31} \\ F_{32}F_{13} - F_{33}F_{12} & F_{33}F_{11} - F_{31}F_{13} & F_{31}F_{12} - F_{32}F_{11} \\ F_{12}F_{23} - F_{13}F_{22} & F_{13}F_{21} - F_{11}F_{23} & F_{11}F_{22} - F_{12}F_{21} \end{bmatrix}, \quad \boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},$$

   where $\boldsymbol{n}$ is a fixed vector (not dependent on $\boldsymbol{u}$), find $D((\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n})(\boldsymbol{u}^0)[\boldsymbol{u}_\Delta]$ in the form $\boldsymbol{R}(\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_c$, where $\boldsymbol{R}$ is the matrix that you have to determine, and

   $$(\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_c = \Big[(\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{11}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{12}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{13}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{21}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{22}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{23}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{31}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{32}\ (\boldsymbol{\nabla}\boldsymbol{u}_\Delta)_{33}\Big]^t.$$

3. Given a vector field $\boldsymbol{v} : V(t) \to \Re^3$ over the deformed configuration $V(t)$, its Piola transform is the vector field $\boldsymbol{u} : V_0 \to \Re^3$ defined over the reference configuration by the relation (30)

   $$\boldsymbol{u}(\boldsymbol{X}) = (\mathbf{cof}\,\boldsymbol{F})^t \boldsymbol{v}(\boldsymbol{x}).$$

   We wish to find a relation between the divergences $\boldsymbol{\nabla}_X \cdot \boldsymbol{u}$ and $\boldsymbol{\nabla}_x \cdot \boldsymbol{v}$. With a view towards this

   (a) If $\boldsymbol{T} \in \text{Lin}$ and $\boldsymbol{a}$ is a vector, find a relation for $\boldsymbol{\nabla} \cdot (\boldsymbol{T}^t \boldsymbol{a})$ in terms of the divergence of $\boldsymbol{T}$ and the gradient of $\boldsymbol{a}$.

   (b) Using this relation and the fact that $\boldsymbol{\nabla}_X \cdot \mathbf{cof}\,\boldsymbol{F} = \boldsymbol{0}$, find the desired relation between $\boldsymbol{\nabla}_X \cdot \boldsymbol{u}$ and $\boldsymbol{\nabla}_x \cdot \boldsymbol{v}$.