Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 26/9/06.
Duration: 3.30 p.m.–5.00 p.m.
Maximum Marks: 100

1. In the class we saw that the space of vectors (3 independent components) is "equivalent" to the space of skew-symmetric tensor (3 independent components), in the sense that for every $\boldsymbol{w} \in V$, there exists $\boldsymbol{W} \in \mathrm{Skw}$ such that $\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u}$ for all $\boldsymbol{u} \in V$, and conversely. This problem explores if a similar equivalence can be set up for symmetric tensors. (20)

   If $\boldsymbol{a}, \boldsymbol{b} \in V$, then obviously $(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})/2 \in \mathrm{Sym}$. Conversely, determine if every $\boldsymbol{S} \in \mathrm{Sym}$ (6 independent components) can be represented as $(\boldsymbol{a}\otimes\boldsymbol{b}+\boldsymbol{b}\otimes\boldsymbol{a})/2$ (3 independent components for $\boldsymbol{a}$ and $\boldsymbol{b}$ each, making a total of 6 independent components). (Hint: Find the principal invariants of $(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})/2$).

2. Let $\boldsymbol{R} \in \mathrm{Orth}^+$, and let $\boldsymbol{e}$ be the axis of $\boldsymbol{R}$, i.e., $\boldsymbol{R}\boldsymbol{e} = \boldsymbol{e}$. Let $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{r}\}$ form an orthonormal basis. (40)

   (a) Express the identity tensor $\boldsymbol{I}$ in terms of the orthonormal basis $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{r}\}$. Substitute this expression in the relation $\boldsymbol{R} = \boldsymbol{R}\boldsymbol{I}$ and derive the following representation of $\boldsymbol{R}$:

   $$\boldsymbol{R} = \boldsymbol{e} \otimes \boldsymbol{e} + \cos\alpha(\boldsymbol{q} \otimes \boldsymbol{q} + \boldsymbol{r} \otimes \boldsymbol{r}) + \sin\alpha(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}),$$

   where $\cos\alpha = (\boldsymbol{R}\boldsymbol{q}) \cdot \boldsymbol{q}$. Justify all your answers *mathematically.*

   (b) Using the above representation, find conditions on α such that $\{\boldsymbol{I}, \boldsymbol{R}, \boldsymbol{R}^2\}$ is a linearly independent set. Also find value/values of α for which the set $\{\boldsymbol{I}, \boldsymbol{R}\}$ (but not $\boldsymbol{R}^2$) is linearly independent, and the value of α for which $\{\boldsymbol{I}, \boldsymbol{R}, \boldsymbol{R}^2\}$ is equivalent to $\{\boldsymbol{I}\}$.

3. Find the directional derivative of $\phi(\boldsymbol{T}) = \det \boldsymbol{T}(\boldsymbol{T}^{-1}) : (\boldsymbol{T}^{-1})$, and use it to find the gradient $\partial\phi/\partial\boldsymbol{T}$. (25)

4. Derive the spatial version of the Piola identity (15)

   $$\boldsymbol{\nabla}_x \cdot (\mathbf{cof}\, \boldsymbol{F}^{-1}) = \mathbf{0}.$$

Some relevant formulae

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d}),$$
$$I_2(\boldsymbol{T}) = \frac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{T})^2 - \mathrm{tr}\,(\boldsymbol{T}^2)\right],$$
$$\det \boldsymbol{T}\,[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{T}\boldsymbol{u}, \boldsymbol{T}\boldsymbol{v}, \boldsymbol{T}\boldsymbol{w}],$$
$$D(\det \boldsymbol{T})[\boldsymbol{U}] = \mathbf{cof}\,\boldsymbol{T} : \boldsymbol{U},$$
$$(\mathbf{cof}\,\boldsymbol{T})_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jpq}T_{mp}T_{nq},$$
$$(\boldsymbol{F}^{-1})_{ij} = \frac{\partial \chi_i^{-1}}{\partial x_j}.$$