Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 25/9/07.
Duration: 3.30 p.m.–5.00 p.m.
Maximum Marks: 100

1. Analogous to the scalar-valued logarithmic function, we define the tensor-valued logarithmic function as $\boldsymbol{T} = \log \boldsymbol{A}$ if $\boldsymbol{A} = e^{\boldsymbol{T}}$. (25)

   (a) Using the fact that $e^{\boldsymbol{T}^T} = (\boldsymbol{e}^{\boldsymbol{T}})^T$, find a relation between $(\log \boldsymbol{A})^T$ and $\log \boldsymbol{A}^T$.

   (b) Using the fact that $(e^{\boldsymbol{T}})^{-1} = e^{-\boldsymbol{T}}$, find an expression for $\log \boldsymbol{A}^{-1}$ in terms of $\log \boldsymbol{A}$. Use this result to find the nature of $\log \boldsymbol{R}$, where $\boldsymbol{R} \in \text{Orth}^+$.

   (c) In class we saw that if $\boldsymbol{W} \in \text{Skw}$, then $e^{\boldsymbol{W}} \in \text{Orth}^+$. Use the result in (b) to find if the converse is true, i.e., given $\boldsymbol{R} \in \text{Orth}^+$, does there always exist a $\boldsymbol{W} \in \text{Skw}$ such that $\boldsymbol{R} = e^{\boldsymbol{W}}$.

2. The exponential of a tensor $\boldsymbol{T}(t) \in \text{Lin}$ is given by the series (25)

$$e^{\boldsymbol{T}(t)} = \boldsymbol{I} + \boldsymbol{T}(t) + \frac{1}{2!}[\boldsymbol{T}(t)]^2 + \cdots,$$

   Prove that

$$\frac{d(e^{\boldsymbol{T}(t)})}{dt} = \dot{\boldsymbol{T}} e^{\boldsymbol{T}(t)} = e^{\boldsymbol{T}(t)} \dot{\boldsymbol{T}},$$

   if and only if $\dot{\boldsymbol{T}}\boldsymbol{T} = \boldsymbol{T}\dot{\boldsymbol{T}}$. (Hint: Note that $e^{\boldsymbol{T}(t)}\boldsymbol{T}(t) = \boldsymbol{T}(t)e^{\boldsymbol{T}(t)}$ and differentiate both sides with respect to t.)

3. Let $\boldsymbol{C} \in \text{Sym}$ be invertible, and let (30)

$$W(\boldsymbol{C}) = \frac{\lambda}{8}(\log \det \boldsymbol{C})^2 + \frac{\mu}{2}(\operatorname{tr} \boldsymbol{C} - 3 - \log \det \boldsymbol{C}).$$

   Using $\boldsymbol{S}(\boldsymbol{C}) = 2\partial W/\partial \boldsymbol{C}$, find an expression for $\boldsymbol{S}(\boldsymbol{C})$. Next, find $\boldsymbol{S}(\boldsymbol{I}) + D\boldsymbol{S}(\boldsymbol{I})[2\boldsymbol{E}]$. (You may first find $\boldsymbol{S}(\boldsymbol{I}) + D\boldsymbol{S}(\boldsymbol{I})[\boldsymbol{U}]$, and then put $\boldsymbol{U} = 2\boldsymbol{E}$).

4. Using the relations $\partial J/\partial \boldsymbol{F} = \textbf{cof}\, \boldsymbol{F}$ and $(\boldsymbol{F}^{-1})_{ij} = \partial \chi_i^{-1}/\partial x_j$, show that (20)

$$\boldsymbol{\nabla}_{\boldsymbol{X}} J = J \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{F}^T.$$