Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require

Date: 22/4/99.
Duration: 9.30 p.m.–12.30 p.m.
Maximum Marks: 200

1. The governing equation for heat conduction through a sphere of radius R is given by (50)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 k\frac{dT}{dr}\right) + Q_0 = 0,$$

where k the thermal conductivity, and Q_0, the heat generated per unit volume, are both constant. The boundary conditions are

$$kr^2\frac{dT}{dr} = 0 \text{ at } r = 0,$$
$$T = T_0 \text{ at } r = R.$$

   (a) Determine the exact solution.

   (b) Formulate the variational formulation (V) in the form $a(T, v) = L(v)\ \forall v$. Is $a(.,.)$ symmetric and positive-definite (justify)?

   (c) Formulate a finite element method based on (V) using 1-D quadratic elements. Find the required matrices as functions of $\boldsymbol{N}$ and $d\boldsymbol{N}/d\xi$. Do *not* carry out the integrations.

   (d) Using one quadratic element, find the temperatures at the center ($r = 0$) and at $r = R/2$. Compare with the exact solution. Use natural coordinates.

2. The horizontal and vertical members of the assembly shown in Fig. 1 are identical in all respects except that their lengths are $2L$ and L, respectively. The vertical member has pin-joints at both its ends. A vertical load P is applied at the end as shown. We are interested in finding the vertical displacement at the point of application of the load. Construct the global stiffness matrix, $\boldsymbol{K}$, and the global load vector $\boldsymbol{f}$ using two elements each of length L for the horizontal member, and one element of length L for the vertical one. Incorporate the boundary conditions using the elimination approach, and write the new $\boldsymbol{K}$ and $\boldsymbol{f}$ matrices. Indicate the displacement degree of freedom that we are interested in finding. Do *not* solve the system of equations to find the required quantity. Assume Young's modulus to be E, moment of inertia to be I and the cross-sectional area to be A. You may use the given element matrices directly. (40)

3. A constant traction p acts normal to the edge 2-3 of a nine-node element as shown in Fig. 2. Assuming the element to be of thickness t, find the consistent load vector, $\int_{\Gamma_t} \boldsymbol{N}^t \bar{\boldsymbol{t}}\, d\Gamma$, corresponding to the degrees of freedom associated with nodes 2, 6 and 3. The coordinates of the nodes 1-9 are $(-1,-1)$, $(1,-1)$, $(1,1)$, $(-1,1)$, $(0,-1)$, $(1.5,0)$, $(0,1)$, $(-1,0)$ and $(0.25,0)$, respectively. Use natural coordinates. (Hint: Express the normal vector in terms of dx/ds and dy/ds.) (60)

4. A time varying distributed load $q(x,t)$ acts on a simply-supported beam as shown in Fig. 3. The governing equation for the transverse vibration of the beam is given by (50)

$$\rho A \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v}{\partial x^2} \right) = q(x,t).$$

   (a) Find the equation and appropriate boundary conditions for determining the undamped natural frequencies of vibration ω, for the continuum problem (do *not* attempt to solve the equation).

   (b) Develop a semi-discrete finite element formulation (for undamped vibrations) of the form $\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$, by first developing the appropriate variational formulation. Present $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{f}$ as functions of the shape function matrix, $\boldsymbol{N}$, and its derivatives with respect to x. Using one element, find the first two natural frequencies and compare them with the first two analytical natural frequencies obtained from

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}.$$

   You may use the given matrices directly.

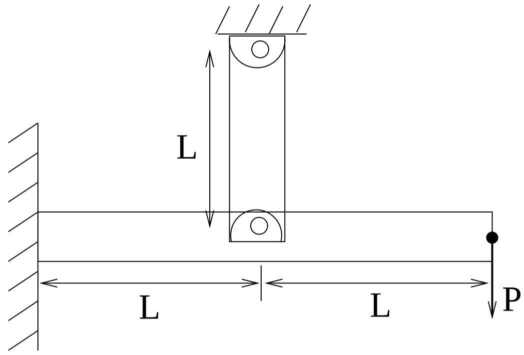

Figure 1:

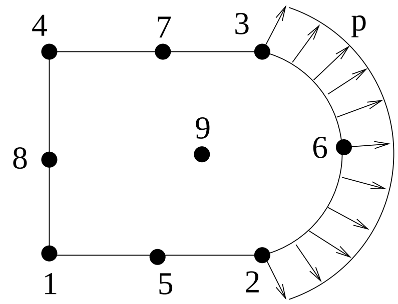

Figure 2:

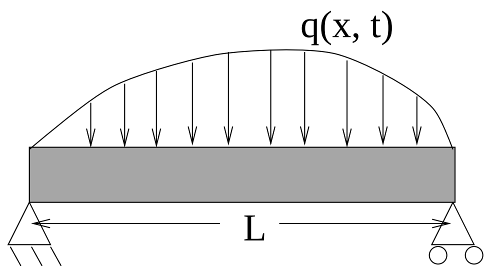

Figure 3:

Some Relevant Formulae

For a linear bar element of length L:

$$\boldsymbol{K}^{(e)} = d\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \text{ where } d = \frac{EA}{L}$$

For a quadratic 1-D element with midnode at the center:

$$\int_{-1}^{1} (1+\xi)^2 \frac{d\boldsymbol{N}^t}{d\xi}\frac{d\boldsymbol{N}}{d\xi}\, d\xi = \begin{bmatrix} 0.4 & -0.8 & 0.4 \\ -0.8 & \frac{64}{15} & -\frac{52}{15} \\ 0.4 & -\frac{52}{15} & \frac{46}{15} \end{bmatrix}$$

$$\int_{-1}^{1} (1+\xi)^2 \boldsymbol{N}^t\, d\xi = \frac{1}{15}\begin{bmatrix} -2 \\ 24 \\ 18 \end{bmatrix}.$$

For a beam element of length L:

$$\boldsymbol{M}^{(e)} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix},$$

$$\boldsymbol{K}^{(e)} = \begin{bmatrix} a & b & -a & b \\ b & 2c & -b & c \\ -a & -b & a & -b \\ b & c & -b & 2c \end{bmatrix},$$

where $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 2EI/L$.