Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end which you can directly use. Derive any other formulae that you may require.

Date: 20/4/2018.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

1. We saw in the test that one of the variational terms was of the type (25)

$$\int_{\Omega} (\boldsymbol{\nabla} \times \boldsymbol{u}_\delta) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \, d\Omega. \tag{1}$$

Consider the axisymmetric case where $\boldsymbol{u} = (u_r(r,z), 0, u_z(r,z))$ so that

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \boldsymbol{e}_\theta.$$

Find the '$\boldsymbol{B}$' matrix that links $\boldsymbol{\nabla} \times \boldsymbol{u}$ to the nodal degrees of freedom of a 4-node quadrilateral element. This $\boldsymbol{B}$ matrix should be expressed in terms of N_i and/or its derivatives $\partial N_i/\partial \xi$ and $\partial N_i/\partial \eta$, $i = 1, 2, 3, 4$. You need not write the shape functions N_i. Express $d\Omega$ in terms of $d\xi \, d\eta$, and finally express the '$\boldsymbol{K}$' matrix corresponding to Eqn. (1) for a single element in terms of $\boldsymbol{B}$.

2. Let Ω denote the domain, and Γ denote the boundary of this domain. The governing equations for thermoelasticity are (35)

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{b} &= \boldsymbol{0} \quad \text{on } \Omega, \\
\boldsymbol{\tau} &= \mathbf{C}\boldsymbol{\epsilon} - 3\kappa\alpha(T - T_0)\boldsymbol{I}, \quad \text{on } \Omega, \\
\boldsymbol{\epsilon} &= \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right] \quad \text{on } \Omega, \\
\boldsymbol{\nabla} \cdot \boldsymbol{q} &= \rho Q_h \quad \text{on } \Omega, \\
\boldsymbol{q} &= -k \boldsymbol{\nabla} T \quad \text{on } \Omega, \\
\boldsymbol{t} &= \bar{\boldsymbol{t}} \quad \text{on } \Gamma_t, \\
\boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_u, \\
T &= \bar{T} \quad \text{on } \Gamma_T, \\
\boldsymbol{q} \cdot \boldsymbol{n} &= -\bar{q} \quad \text{on } \Gamma_q.
\end{aligned}$$

Assume the material properties $\mathbf{C}$, κ, α etc., and the ambient temperature T_0 to be given. Using indicial notation, derive any tensorial identities that you need along the way.

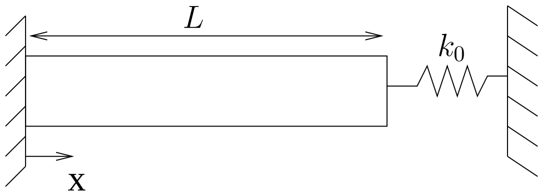

Figure 1: Rod subjected to prescribed temperature $T = T_1$ at $x = 0$ and prescribed normal flux of zero at $x = L$.

(a) By operating on the linear momentum balance and thermal balance equations by the variations of the displacement and temperature, respectively, and carrying out an appropriate integration by parts, find the (coupled) variational forms for determining the displacement and temperature fields.

(b) Discretize the displacement and temperature fields as

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{N}_u \hat{\boldsymbol{u}},\\ T &= \boldsymbol{N}_T \hat{T},\\ \boldsymbol{\epsilon} &= \boldsymbol{B}_u \hat{\boldsymbol{u}},\\ \boldsymbol{\nabla} T &= \boldsymbol{B}_T \hat{T}.\end{aligned}$$

(without giving the details of what $\boldsymbol{N}_u$, $\boldsymbol{B}_u$ etc. are), and develop the finite element matrix equations in the form

$$\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uT} \\ \boldsymbol{K}_{Tu} & \boldsymbol{K}_{TT} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{T} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_T \end{bmatrix},$$

where $\boldsymbol{K}_{uu}$, $\boldsymbol{K}_{TT}$, $\boldsymbol{f}_u$ etc. are to be expressed in terms of $\boldsymbol{N}_u$, $\boldsymbol{N}_T$, $\boldsymbol{B}_u$ etc.

(c) Consider the setup in Fig. 1 where the rod has a cross sectional area of A. By considering the 1D approximations $\epsilon = \partial u/\partial x$, $q = -k\partial T/\partial x$, $\tau = E[\epsilon - \alpha(T - T_0)]$, $\mathbf{C} \equiv E$, $3\kappa \equiv E$, etc., and using linear shape functions for both the displacement and the temperature (i.e., $N_u = N_T$), find the displacement and temperature at the end of the bar, i.e., at $x = L$, and the stress in the bar if a temperature of $T = T_1$ is imposed at the left end, and a normal flux of zero is imposed at $x = L$. Your answers should obviously be in terms of known parameters such as E, A, T_1 etc. You may incorporate the effect of the spring through the traction term or otherwise. Assume T_0, A, and the material properties to be constant. You may directly use the matrices at the back, and derive any matrices that are not given there.

3. The governing equation for a beam of length L subjected to a distributed load per unit length $q(x,t)$ as shown in Fig. 2 is given by (40)

$$\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) = q(x,t), \qquad (2)$$

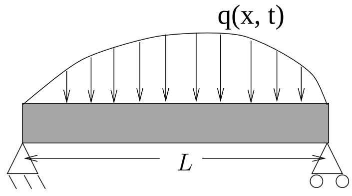

Figure 2: Beam subjected to a distributed load $q(x, t)$.

where ρ is the density, A is the cross sectional area, I is the moment of inertia, E is the Young modulus, and $w(x, t)$ is the transverse displacement.

(a) Starting from the beam equation given by Eqn. (2), derive conservation laws analogous to linear momentum and energy assuming traction-free conditions on the entire beam structure. *All your operations should be with Eqn.* (2) *as a starting point, i.e., do not use terms such as* $\int_V \rho \boldsymbol{v}\, dV$ *etc. as we did in class*; an equation analogous to angular momentum conservation can also be derived, but we will keep the exam simple by ignoring that :-). (Hint: Multiply Eqn. (2) with the appropriate velocity for deriving energy conservation).

(b) Develop the variational formulation corresponding to Eqn. (2) for the problem shown in Fig. 2.

(c) By discretizing as

$$\boldsymbol{w} = \boldsymbol{N}\hat{\boldsymbol{w}},$$
$$\frac{\partial^2 \boldsymbol{w}}{\partial x^2} = \boldsymbol{B}\hat{\boldsymbol{w}},$$

where $\boldsymbol{N}$ is the matrix of Hermite shape functions, and $\boldsymbol{B} = \partial^2 \boldsymbol{N}/\partial x^2$, and assuming a single element mesh, find the semidiscrete form of the finite element equations with $\boldsymbol{M}$ and $\boldsymbol{K}$ expressed in terms of $\boldsymbol{N}$ and $\boldsymbol{B}$ (*do not* present the expressions for the elements of $\boldsymbol{N}$ or $\boldsymbol{B}$ in terms of Hermite shape functions.)

(d) Develop an energy-momentum conserving algorithm on the time interval $[t_n, t_{n+1}]$ based on the semi-discrete form. Prove the conservation properties that mimic those of the continuum that you derived in part (a) above.

(e) Show how you will incorporate the boundary conditions for the problem in Fig. 2. Take the entries of $\boldsymbol{K}$ as K_{11}, K_{12} etc. without presenting expressions for them.

(f) For the problem shown in Fig. 2, imagine $q(x, t)$ is suddenly set to zero at some time T, but with the supports still there. Is the (a) linear momentum (b) total energy, that you derived in part (a) above conserved (both for the continuum and finite element approximations) from time T onwards? Justify your answers.

Some Relevant Formulae

For a linear bar element of length L:

$$\boldsymbol{N}_u = \boldsymbol{N}_T = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix},$$

$$\boldsymbol{B}_u = \boldsymbol{B}_T = \frac{d\boldsymbol{N}}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix},$$

$$\int_0^L \frac{d\boldsymbol{N}^T}{dx}\frac{d\boldsymbol{N}}{dx}\,dx = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$