Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Date: 19/4/2024.
Duration: 9.15 a.m.–12.15 p.m.
Maximum Marks: 100

1. A hollow circular cylinder of inner radius a and outer radius b, and of length L is acted upon by a tangential traction $t_\theta(z)$ on the inner surface $r = a$. The bottom and top surfaces $z = 0$ and $z = L$ are fixed. Assume that $u_z = c_1 + c_2 r + c_3 z$, and $u_\theta(r, z)$ are the only nonzero components of the displacement vector. (30)

   (a) Determine the constants c_1, c_2 and c_3.

   (b) Simplify the following relations

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \epsilon_{r\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right], \\
\epsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \\
\boldsymbol{\tau} &= \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon},
\end{aligned} \tag{1}$$

   where the Lame parameters (λ, μ) are constant.

   (c) Let V denote the domain, and S_t denote the part of the surface on which tractions are applied. Specialize the variational formulation

$$\int_V \boldsymbol{\epsilon}(\boldsymbol{v}) : \boldsymbol{\tau}\, dV = \int_{S_t} \boldsymbol{v} \cdot \bar{\boldsymbol{t}}\, dS \quad \forall \boldsymbol{v}, \tag{2}$$

   to the problem at hand, i.e., write this equation in terms of the individual stress, strain and traction components in the context of an axisymmetric formulation.

   (d) Discretize the nonzero displacement components in your formulation using the 4-node quadrilateral shape functions (assume the quadrilateral element to be distorted and not necessarily a rectangle; you need not state the shape functions in terms of (ξ, η)). Denoting the inverse of the Jacobian by $\boldsymbol{J}^{-1} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$, derive the strain-displacement matrix $\boldsymbol{B}$ in terms of H_{ij}, N_i, $\partial N_i/\partial \xi$ and $\partial N_i/\partial \eta$.

   (e) State the element stiffness matrix $\boldsymbol{K}$, and the element load vector $\boldsymbol{f}$ in terms of $\boldsymbol{B}$ and the shape function matrix $\boldsymbol{N}$. *Do not* carry out the integrations in the expressions for $\boldsymbol{K}$ and $\boldsymbol{f}$, but state them with the proper integration limits in the natural coordinate system.

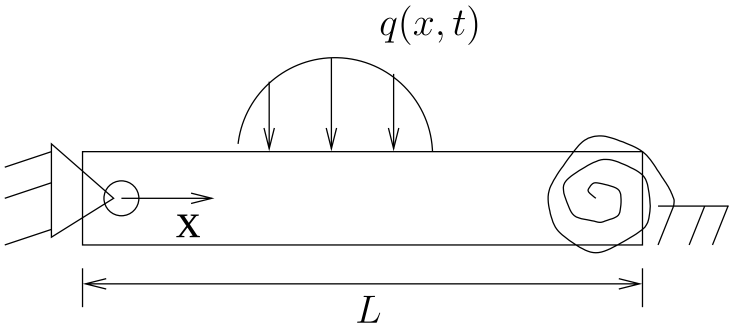

Figure 1: Beam subjected to a distributed load $q(x,t)$, and with a torsional spring at the right end.

(f) If $t_\theta|_{r=a} = t_0$, where t_0 is a constant, find an explicit expression for the consistent load vector for a single element mesh (which is of course rectangular).

2. The governing equation for a beam of length L subjected to a distributed load per unit length $q(x,t)$ as shown in Fig. 1 is given by (40)

$$\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = q(x,t), \tag{3}$$

where ρ is the density, A is the cross sectional area, I is the moment of inertia, E is the Young modulus, and $w(x,t)$ is the transverse displacement. The boundary condition associated with the torsional spring of stiffness k is given by

$$\left[EI\frac{\partial^2 w}{\partial x^2} + k\frac{\partial w}{\partial x}\right]_{x=L} = 0.$$

(a) Denoting the variation of w by w_δ, develop the variational formulation corresponding to Eqn. (3) for the problem shown in Fig. 1.

(b) By choosing the variation w_δ to be an appropriate velocity (which is equivalent to multiplying the governing equation by an appropriate velocity and carrying out an integration by parts since the variation and the velocity satisfy the same homogeneous boundary conditions), derive an energy conservation law for the setup shown in Fig. 1 when $q(x,t) = 0$.

(c) By discretizing the variational formulation in part (a) as

$$\boldsymbol{w} = \boldsymbol{N}\hat{\boldsymbol{w}},$$
$$\frac{\partial^2 \boldsymbol{w}}{\partial x^2} = \boldsymbol{B}\hat{\boldsymbol{w}},$$

where $\boldsymbol{N}$ is the matrix of Hermite shape functions, and $\boldsymbol{B} = \partial^2\boldsymbol{N}/\partial x^2$, and assuming a single element mesh, find the semidiscrete form of the finite element equations with $\boldsymbol{M}$ and $\boldsymbol{K}$ expressed in terms of $\boldsymbol{N}$ and $\boldsymbol{B}$ (*do not* present the expressions for the elements of $\boldsymbol{N}$ or $\boldsymbol{B}$ in terms of Hermite shape functions, and let all integrals be with respect to dx.)

(d) Develop an energy-conserving algorithm on the time interval $[t_n, t_{n+1}]$ based on the semi-discrete form. Prove the conservation properties that mimic those of the continuum that you derived in part (b) above.

(e) For solving the above problem with one beam element, which degrees of freedom will you suppress, and which are the free degrees of freedom that you will solve for?

(f) Imagine $q(x,t)$ is suddenly set to zero at some time T, but with the supports and torsional spring still present. Is the total energy, that you derived in part (b) above conserved (both for the continuum and finite element approximations) from time T onwards? Justify your answers.

3. The radially symmetric transient heat conduction equation for the temperature field $T(r,t)$ in a hollow circular disc with material properties (ρ_1, c_1, k_1) and (ρ_2, c_2, k_2) in the regions $r \in [r_1, r_2]$ and $r \in [r_2, r_3]$, respectively, is given by (30)

$$\rho_i c_i \frac{\partial T}{\partial t} = \frac{k_i}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right), \quad i = 1, 2. \tag{4}$$

A normal flux $(k_1 \partial T/\partial r)_{r=r_1} = q_0(t)$ acts at the inner surface, and the temperature at the outer surface is given by $T|_{r=r_3} = T_0(t)$. Assume throughout this problem that the temperature is a function only of the radial coordinate and time, i.e., $T = T(r,t)$.

(a) Develop the semi-discrete formulation corresponding to Eqn. (4) and the above boundary conditions. For a two-node linear element, present the expressions for the element level matrices $\boldsymbol{M}^{(e)}$, $\boldsymbol{K}^{(e)}$ and $\boldsymbol{f}^{(e)}$ with respect to the natural coordinate system. Do not carry out the integrations in the expressions for $\boldsymbol{M}^{(e)}$ and $\boldsymbol{K}^{(e)}$, but present explicit expressions for $\boldsymbol{f}$ (in particular, what is the load value at the interface node?). For a mesh of two linear elements, find the global load vector $\boldsymbol{f}$. *Justify* the value of the load vector component at the interface node.

(b) Denoting the global matrices as $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{f}$, develop a fully discrete formulation using the backward Euler method ($\dot{\hat{\boldsymbol{T}}} = (\hat{\boldsymbol{T}}_{n+1} - \hat{\boldsymbol{T}}_n)/t_\Delta$). Denoting the entries in $\boldsymbol{M}$ and $\boldsymbol{K}$ as $\begin{bmatrix} M_{11} & M_{12} & \dots \\ M_{12} & M_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}$, and $\begin{bmatrix} K_{11} & K_{12} & \dots \\ K_{12} & K_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}$, etc., respectively, show how you will incorporate the boundary conditions in the global set of equations. Given that the initial temperature is zero, present a strategy for finding the temperature at time t_Δ.