Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Date: 23/4/2025.
Duration: 9.00 a.m.–12.15 p.m.
Maximum Marks: 100

1. Treat this entire problem as a statics problem . Consider the setup shown in Fig. 1a, where the end points of two beams both of length L and with identical properties E, I, etc., are separated by a distance Δ. The end point of the upper beam is connected to a spring with spring constant k; the spring is undeformed in this original configuration as shown in Fig. 1a. The tip of the lower beam is pulled up until it is in contact with the tip of the upper beam, and then connected to it using a pin joint. Then the system is released so that it reaches equilibrium as shown in Fig. 1b. (35)

   (a) Modify the potential energy for a beam given by

   $$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L qv\,dx - \sum P_i v_i - \sum M_i v_i',$$

   so as to account for the presence of the spring, and for the multipoint constraint linking the displacements of the two beams (Note: This problem is to be solved using the multi-point constraint approach only).

   (b) By taking the first variation of the modified potential energy, and discretizing the displacement field as $v = \boldsymbol{N}\hat{\boldsymbol{u}}$ (you *need not* write the individual elements of $\boldsymbol{N}$), develop the finite element formulation for solving this problem. State the stiffness matrix in terms of derivatives of $\boldsymbol{N}$.

   (c) Using one beam element each for the top and bottom beams, write the global (assembled) matrix system in the form $\boldsymbol{K}\hat{\boldsymbol{u}} = \boldsymbol{f}$. You can

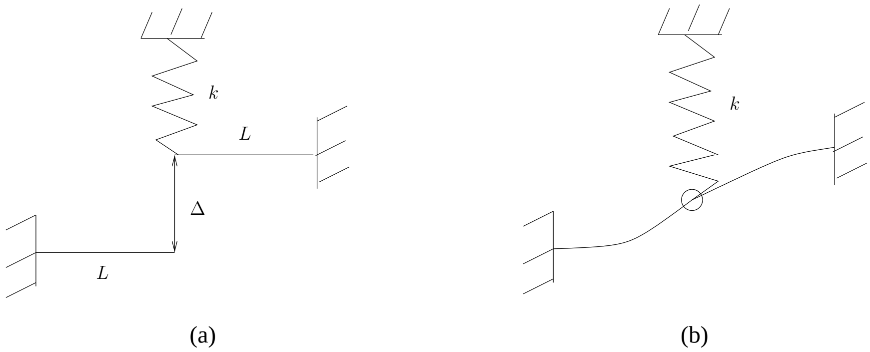

Figure 1: Problem 1

directly use the element stiffness matrix given at the end (without substituting for a, b etc.). Incorporate the boundary conditions, and write the complete system of equations to be solved for the free degrees of freedom. *Do not attempt to solve this system of equations.*

2. The equation for Couette-type flow (you don't need any knowledge of fluid mechanics to solve this problem; also do not use the results from the notes directly since you are being asked to prove certain things in this problem), with $u = u(y)$ for $y \in [0, h]$, and $\nu > 0$, is (30)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{1}$$

The boundary conditions are $u|_{y=0} = 0$ and $(\nu \partial u/\partial y)_{y=h} = \bar{p}(t)$, and the initial condition is $u|_{t=0} = cy/h$, where c is a real-valued constant (positive or negative).

   (a) Develop the variational formulation corresponding to the above equation.

   (b) By discretizing the variational formulation using a single two-node linear element, develop the semi-discrete formulation corresponding to Eqn. (1), and the above boundary conditions. You may directly use the results at the back for deriving your $\boldsymbol{M}$ and $\boldsymbol{K}$ matrices.

   (c) Incorporate the essential boundary condition into your semi-discrete form.

   (d) Using the generalized trapezoidal rule, develop the full-discrete version of the semi-discrete equation in the previous part on the time interval $[t_n, t_{n+1}]$. Eliminate v_{n+1} so that we can solve the fully-discrete version for u_{n+1}.

   (e) Now let $\bar{p} = 0$. Derive the growth or decay property for the variable in your semi-discrete formulation by solving the differential equation.

   (f) By mimicking this growth or decay property in your fully-discrete formulation, find the values for α that lead to an unconditionally stable algorithm. For a conditionally stable algorithm, derive the restriction on the time step t_Δ.

3. The governing equation for the transverse displacement $w(x, y, t)$ of a membrane whose domain is denoted by V (which is not necessarily rectangular) subjected to a distributed load $q(x, y, t)$ is given by (35)

$$\rho \frac{\partial^2 w}{\partial t^2} = T \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + q(x, y, t), \quad \text{in } V, \tag{2}$$

where T is the given tension, and ρ is the density.

   (a) Denoting the variation of w by w_δ, develop the variational formulation corresponding to Eqn. (2), and identify the allowable boundary conditions.

(b) By choosing the variation w_δ to be an appropriate velocity (which is equivalent to multiplying the governing equation by an appropriate velocity and carrying out an integration by parts since the variation and the velocity satisfy the same homogeneous boundary conditions), derive an energy conservation law for the membrane when $q(x, y, t) = 0$, and when the applied load along the entire boundary is zero.

(c) By discretizing the variational formulation in part (a) as

$$w = \boldsymbol{N}\hat{\boldsymbol{w}},$$

where $\boldsymbol{N}$ is the matrix of shape functions for a $2D$ element, find the semidiscrete form of the finite element equations with $\boldsymbol{M}$ and $\boldsymbol{K}$ expressed in terms of $\boldsymbol{N}$ and $\boldsymbol{B}$ (with $\boldsymbol{B}$ expressed in terms of partial derivatives of $\boldsymbol{N}$ using indicial notation)

(d) Assuming appropriate initial conditions, develop an energy-conserving algorithm on the time interval $[t_n, t_{n+1}]$. Prove the energy-conservation property that mimics that of the continuum that you derived in part (b) above.

(e) If $w = 0$ on the entire boundary of the membrane, and if $q(x, y, t)$ is suddenly set to zero at some time T, will the total energy, that you derived in part (b) above, be conserved (both for the continuum and finite element approximations) from time T onwards? Justify your answers.

(f) Now consider the domain to be a circle of radius 2, with $q(x, y, t)$ equal to zero, and with $T(\partial w/\partial r)_{r=2} = \bar{t}(1 - e^{-t})$, where $\bar{t}$ is a constant. Considering the symmetry in the problem, we discretize only a quarter of the domain with a 6-node triangular element with the nodal coordinates given by $(x_1, y_1) = (2, 0)$, $(x_2, y_2)_2 = (0, 2)$, $(x_3, y_3)_3 = (0, 0)$, $(x_4, y_4) = (\sqrt{2}, \sqrt{2})$, $(x_5, y_5) = (0, 1)$ and $(x_6, y_6) = (1, 0)$. Find the consistent load vector corresponding to the given natural boundary condition (*Do not* evaluate any complicated integrals that arise; however, state the proper integration limits and the integrand). While incorporating the boundary conditions for the 6-node triangular element above, what are the nodal variables (if any) that you will suppress (in other words, what is the size of the $\boldsymbol{K}$ and $\boldsymbol{M}$ matrices *after* incorporating the boundary conditions).

For a two-node element,

$$\int_0^h \boldsymbol{N}^T \boldsymbol{N}\, dy = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\int_0^h \frac{d\boldsymbol{N}}{dy}^T \frac{d\boldsymbol{N}}{dy}\, dy = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For a beam element of length L:

$$\boldsymbol{K}^{(e)} = \begin{bmatrix} a & b & -a & b \\ b & 2c & -b & c \\ -a & -b & a & -b \\ b & c & -b & 2c \end{bmatrix},$$

where $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 2EI/L$.

Shape functions for a 6-node triangular element:

$$
\begin{aligned}
N_1 &= \xi(2\xi - 1), & N_4 &= 4\xi\eta, \\
N_2 &= \eta(2\eta - 1), & N_5 &= 4\alpha\eta, \\
N_3 &= \alpha(2\alpha - 1), & N_6 &= 4\xi\alpha,
\end{aligned}
$$

where $\alpha = (1 - \xi - \eta)$.