Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Date: 22/4/2026.
Duration: 9.00 a.m.–12.30 p.m.
Maximum Marks: 100

1. Treat this entire problem as a statics problem. Consider the setup shown in Fig. 1, where the vertical bar of length L is initially separated from the horizontal simply-supported beam of length $2L$ by a distance Δ. The vertical bar is heated by an amount T_Δ. Assume that T_Δ is large enough that the vertical bar makes contact with the horizontal beam exactly at its center, and deflects it. The Young modulus and cross sectional area are E and A for both members. (35)

   (a) The one-dimensional variational formulation for the displacement $u(x)$ in a bar of uniform cross-section with area A, length L, coefficient of thermal expansion α, and Young modulus E, which is subjected to a constant temperature change T_Δ in the absence of body and surface forces, is given by
   $$\int_0^L \frac{dv}{dx}\tau(x)\, A\, dx = 0 \quad \forall v.$$
   Using the relation $\tau = E(du/dx - \alpha T_\Delta)$, develop the element stiffness matrix and load vector for a two-node bar element.

   (b) Exploit symmetry and discretize the symmetric half that you are modeling using two elements.

   (c) Impose any constraints on the displacement that involve two or more degrees of freedom using a Lagrange multiplier technique (do not directly use the results from the notes but show how you obtain your equations by writing the variational formulation associated with the constraint equation). Using the notation $d := EA/L$, form the global system of equations corresponding to these two elements which includes the Lagrange multiplier associated with the constraint.

   (d) Incorporate the homogeneous boundary conditions in this global system of equations.

   (e) Using your matrix equations, indicate (without explicitly solving) how you would solve for the transverse displacement at the center of the horizontal member, and the reaction at the top support.

2. A composite sphere of radius a with properties ρ_1, c_1 and k_1 in the region $r \leq a/2$, and properties ρ_2, c_2 and k_2 in the region $a/2 \leq r \leq a$, where r denotes the spherical radial coordinate is subjected to a normal flux $(k_2 \partial T/\partial r)_{r=a} =$ (30)

$q_0(t)$. In terms of the spherical coordinates (r, θ, ϕ), where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, the governing equation is given by

$$\rho_i c_i \frac{\partial T}{\partial t} = k_i \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right], \quad i = 1, 2. \tag{1}$$

(a) Taking into account the spherical nature of the domain, and the nature of the loading, state the variables on which T is dependent (e.g. $T = T(\phi)$). Using this dependence, write the reduced form of the governing equation given in Eqn. (1).

(b) Develop the variational formulation for your reduced equation.

(c) Discretize the domain with either two linear elements or one quadratic element with nodes at $r = 0, a/2, a$. *This choice should be based on the physics of the problem. Provide a rigorous justification for your choice.*

(d) Depending on your choice, develop the element level matrices $\boldsymbol{M}^{(e)}$, $\boldsymbol{K}^{(e)}$ and $\boldsymbol{f}^{(e)}$ using natural coordinates. State your matrices in terms of integrals. *Do not evaluate these integrals.* However, state your global load vector explicitly with proper justifications (especially at the node at $r = a/2$).

(e) Denoting the global matrices as $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{f}$, develop a fully discrete formulation using the backward Euler method ($\alpha = 1$)

(f) In case any boundary conditions are to be incorporated in the global set of equations, denote the entries in $\boldsymbol{M}$ and $\boldsymbol{K}$ as $\begin{bmatrix} M_{11} & M_{12} & \dots \\ M_{12} & M_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}$, and $\begin{bmatrix} K_{11} & K_{12} & \dots \\ K_{12} & K_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}$, etc., respectively. In case, no boundary conditions are to be incorporated, skip this step.

(g) Given that the initial temperature is zero, present a strategy for finding the nodal temperatures at the first time step t_Δ based on your backward Euler method.

3. The governing equation for the transverse displacement $w(x, y, t)$ of a membrane whose domain is denoted by V (which is not necessarily rectangular) subjected to a distributed load $q(x, y, t)$, and which rests on an elastic foundation is given by (35)

$$\rho \frac{\partial^2 w}{\partial t^2} = T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - kw + q(x, y, t), \quad \text{in } V, \tag{2}$$

where T is the given tension, ρ is the density, and k is a non-negative constant; $k = 0$ corresponds to the case where the elastic foundation is absent.

(a) Denoting the variation of w by w_δ, develop the variational formulation corresponding to Eqn. (2), and identify the allowable boundary conditions.

(b) By choosing w_δ to be an appropriate quantity, derive a law that is analogous to linear momentum conservation. Clearly state the conditions that should hold for this linear momentum conservation law to be valid (e.g., $q = 0$ etc.)

(c) By choosing the variation w_δ to be an appropriate velocity (which is equivalent to multiplying the governing equation by an appropriate velocity and carrying out an integration by parts since the variation and the velocity satisfy the same homogeneous boundary conditions), derive an energy conservation law for the membrane. Again state the conditions under which this energy conservation equation is valid (e.g., $q = 0$, etc.). (Hint: Multiple options may work.)

(d) By discretizing the variational formulation in part (a) as

$$w = \boldsymbol{N}\hat{\boldsymbol{w}},$$

where $\boldsymbol{N}$ is the matrix of shape functions for a $2D$ element, find the semidiscrete form of the finite element equations with $\boldsymbol{M}$ and $\boldsymbol{K}$ expressed in terms of $\boldsymbol{N}$ and $\boldsymbol{B}$ (with $\boldsymbol{B}$ expressed in terms of partial derivatives of $\boldsymbol{N}$ with respect to x and y)

(e) Assuming appropriate initial conditions, develop an energy-momentum conserving algorithm on the time interval $[t_n, t_{n+1}]$. Prove the energy and momentum conservation properties that mimic those of the continuum that you derived in parts (b) and (c) above.

(f) If the domain is circular with unit radius, and if the product of the natural boundary condition and T has a constant value of 1, find the consistent load vector corresponding to the three nodes of a Q9 element that lie on the circumference, and whose coordinates are $(1, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$ and $(0, 1)$. You can state the result in the form of integrals with respect to the natural coordinate ξ, and which you do not need to evaluate. You may directly use the 1D shape functions $N_1 = -\xi(1-\xi)/2$, $N_2 = 1-\xi^2$ and $N_3 = \xi(1+\xi)/2$ on this edge, and state the load vector only for these three nodes.

Relevant formulae

For a beam element of length L:

$$\boldsymbol{K}^{(e)} = \begin{bmatrix} a & b & -a & b \\ b & 2c & -b & c \\ -a & -b & a & -b \\ b & c & -b & 2c \end{bmatrix},$$

where $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 2EI/L$.

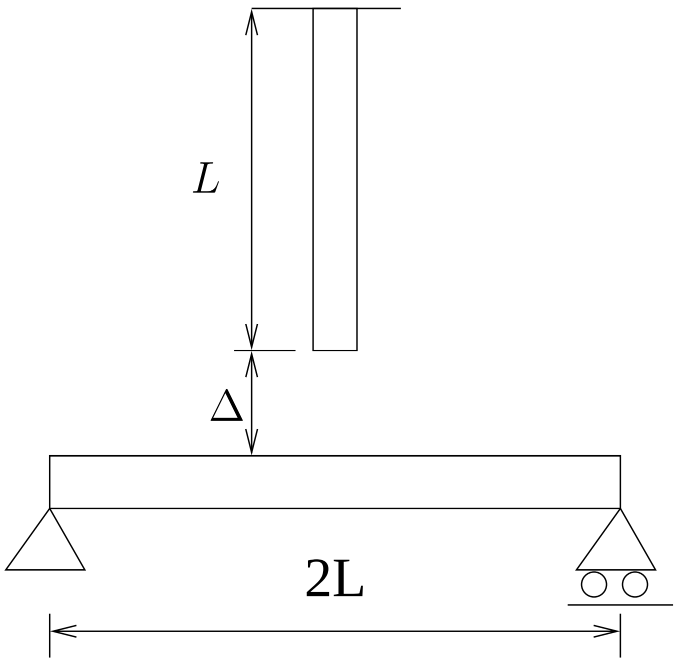

Figure 1: Problem 1