Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require.

Date: 20/4/2002.
Duration: 2.30 p.m.–5.30 p.m.
Maximum Marks: 200

1. You may directly use the matrices given to solve the following sub-problems. All members may be assumed to have the same properties such as E, A, I etc. (50)

   (a) Exploit symmetry and use one element to find the deflection under the load for the structures shown in Figs. 1a and 1b. The joint in Fig. 1b is a frictionless pin joint.

   (b) The vertical member in Fig. 1c is subjected to a temperature change ΔT. Find the vertical deflection at the pin joint, and the reaction at the support of the vertical member, by using one element to model each member.

2. The governing equation for the torsion stress function Ψ, for a bar of uniform cross-section which is subjected to a torque M, is given by (70)

$$\boldsymbol{\nabla}^2\Psi + 2 = 0 \text{ in } \Omega,$$
$$\Psi = 0 \text{ on } \Gamma,$$

where Ω is the cross-section of the bar, and Γ constitutes the boundary of Ω. Develop the variational formulation corresponding to the above governing equation and boundary condition, and use it to develop an isoparametric formulation for a *rectangular* Q4 element whose coordinates are given by (x_1, y_1), (x_2, y_1), (x_2, y_2) and (x_1, y_2). You may directly use the relations

$$x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2,$$
$$y = \frac{1-\eta}{2}y_1 + \frac{1+\eta}{2}y_2,$$

which are valid for this geometry. Present your expressions for the stiffness matrix $\boldsymbol{K}$ and the force matrix $\boldsymbol{f}$ in terms of natural coordinates (use the notation $a \equiv x_2 - x - 1$ and $b \equiv y_2 - y_1$). Evaluate *only* K_{11} and f_1. Use these values to solve the problem shown in Fig. 2; by exploiting symmetry, discretize only the top upper quarter of the domain using one element with the node numbering as shown, find the value of Ψ at node 1, and use this value to find the quantity

$$\frac{M}{2G\alpha} = \int_{\Omega} \Psi \, d\Omega,$$

(using natural coordinates) for the *entire* domain.

3. For the torsion of a circular cylinder of length L which is fixed at one end and subjected to a time-varying moment $M(t)$ at the other, the displacement field is given by (80)

$$u_x = -y\phi(z,t); \quad u_y = x\phi(z,t); \quad u_z = 0,$$

where the z-axis is directed along the axis of the cylinder, and ϕ denotes the angular displacement.

(a) Using the stress-strain relation $\tau_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij}$, where $\epsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ and $\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$, find the expressions for the stresses. Substituting these relations into the variational formulation

$$-\int_{\Omega} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \boldsymbol{v} \, d\Omega + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} \, d\Omega + \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{v} \, d\Gamma - \int_{\Omega} \boldsymbol{\tau} : \boldsymbol{\epsilon}(\boldsymbol{v}) \, d\Omega = 0 \quad \forall \boldsymbol{v},$$

derive the governing equation for ϕ. Assume body forces to be zero, and use the fact that $\int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{v} \, d\Gamma = M\beta|_{z=L}$, where β denotes the variation of ϕ. The governing equation should involve the polar moment of inertia, $J = \int_A (x^2 + y^2) \, dA$, where A denotes the cross-section of the cylinder (*do not* find an explicit expression for J), and integrals of the type $\int_0^L ...dz$; thus, effectively the problem is reduced to a one-dimensional problem.

(b) By discretizing this one dimensional equation, develop a semi-discrete finite element formulation (for undamped vibrations) of the form $\boldsymbol{M}\ddot{\hat{\boldsymbol{\phi}}} + \boldsymbol{K}\hat{\boldsymbol{\phi}} = \boldsymbol{f}$. Present $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{f}$ as functions of the shape function matrix, $\boldsymbol{N}$, and its derivatives with respect to z. Using one quadratic element with midnode at the center, and assuming uniform J, ρ etc. along the axis of the cylinder, find the governing algebraic equation for determining the first two natural frequencies (*do not* solve this equation). You may use the given matrices directly.

(c) Let $M(t) = M_0 \cos \hat{\omega} t$, where $\hat{\omega}$ is given. Assuming the steady-state solution to be given by $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}_0 \cos \hat{\omega} t$, determine $\hat{\boldsymbol{\phi}}_0$. What happens when $\hat{\omega}$ equals any of the natural frequencies?

(d) Show using *mathematical equations* how the set of equations $\boldsymbol{M}\ddot{\hat{\boldsymbol{\phi}}} + \boldsymbol{K}\hat{\boldsymbol{\phi}} = \boldsymbol{f}$ can be uncoupled. *Do not* actually carry out the uncoupling.

Some Relevant Formulae

For a linear bar element of length L:

$$\boldsymbol{K}^{(e)} = d \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \text{ where } d = \frac{EA}{L},$$

$$\boldsymbol{f}^{(e)} = EA\alpha\Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For a quadratic 1-D element of length L with midnode at the center:

$$\int_0^L \boldsymbol{N}^t \boldsymbol{N}\, dz = \frac{L}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\int_0^L \frac{d\boldsymbol{N}^t}{dz} \frac{d\boldsymbol{N}}{dz}\, dx = \frac{1}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}.$$

For a beam element of length L:

$$\boldsymbol{K}^{(e)} = \begin{bmatrix} a & b & -a & b \\ b & 2c & -b & c \\ -a & -b & a & -b \\ b & c & -b & 2c \end{bmatrix},$$

where $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 2EI/L$.

Shape functions for a 4-node quadrilateral element:

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i).$$

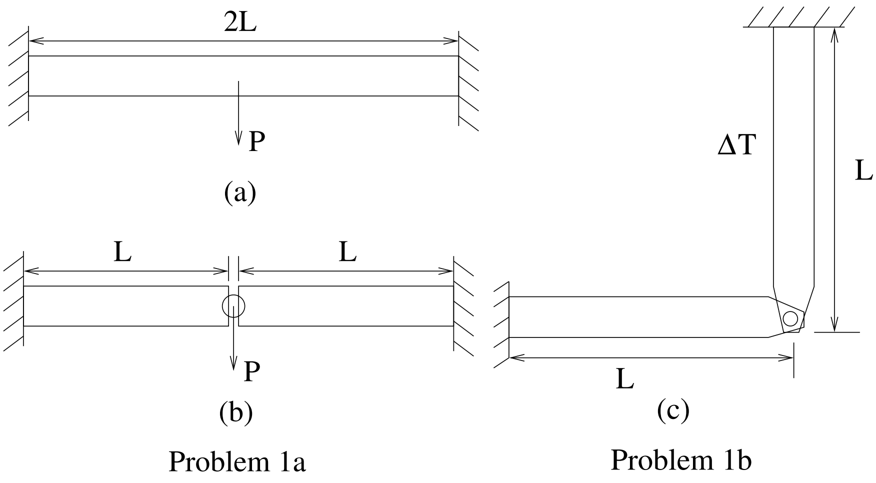

Figure 1:

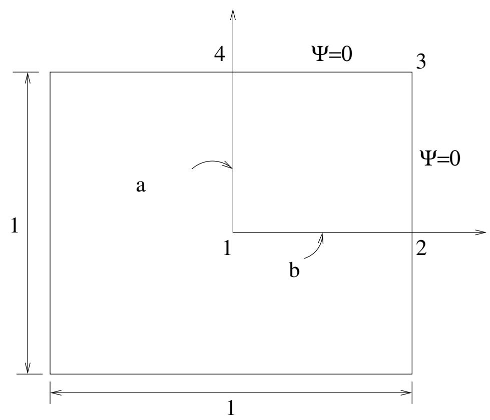

Figure 2: Problem 2