Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require.

Date: 29/4/2010.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. We saw in the test that the governing equation and boundary condition for the torsion warping function are given by (35)

$$\boldsymbol{\nabla}^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \quad \text{in } A,$$
$$(\boldsymbol{\nabla}\psi)\cdot\boldsymbol{n} = yn_x - xn_y \quad \text{on } C.$$

As in the test, starting with the statement $\int_A \phi\boldsymbol{\nabla}^2\psi\, dA = 0$ (where ϕ is the variation of ψ), formulate the variational formulation for the above problem. *The variational formulation should involve only integrals over the domain A.* The triangular domain shown in Fig. 1 is discretized using one triangular element with three nodes as shown. The axis of torsion is through the origin, i.e., the value of ψ at node 3 is zero. *Using natural coordinates*, develop the element stiffness matrix and load vector for the given problem, and find the values of ψ at the remaining two nodes. The element stiffness matrix and load vector can be formulated directly for the element shown in the Figure, and *need not be* for a generic element with arbitrary coordinates. The chain rule if used should have all the terms stated even though some of them could be zero.

2. The axisymmetric domain of square cross section shown in Fig. 2 is subjected (35)

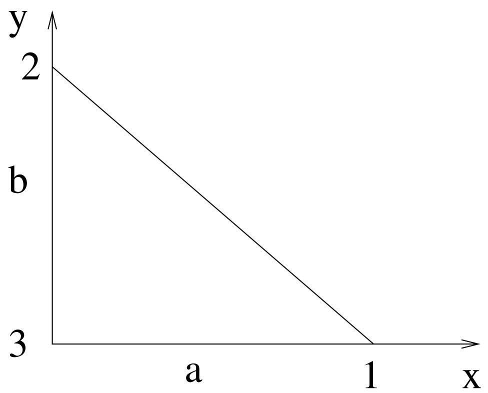

Figure 1: Problem 1

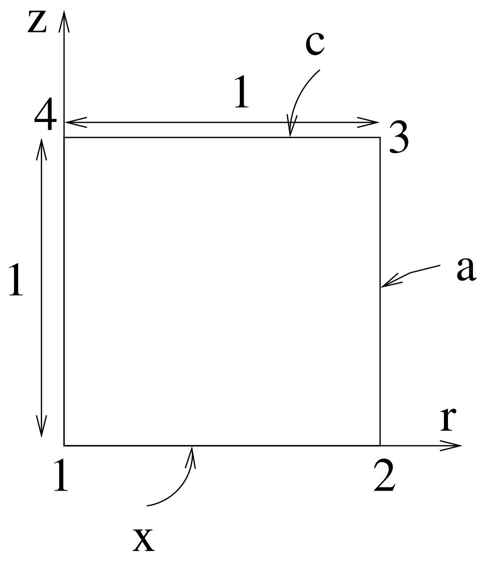

Figure 2: Problem 2

to a heat flux $\bar{q}$ on the bottom surface. The top surface is maintained at a constant temperature $T = 0$. The governing equation for heat conduction is $k\boldsymbol{\nabla}^2 T = 0$, where k denotes the thermal conductivity, and

$$\boldsymbol{\nabla}^2 T \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}.$$

The boundary conditions are as shown.

(a) Develop the variational formulation (Note: $\boldsymbol{\nabla} T \equiv (\partial T/\partial r, \partial T/\partial z)$, $k\boldsymbol{\nabla} T \cdot \boldsymbol{n} = \bar{q}$, the prescribed flux).

(b) Develop the expressions for the stiffness and load vectors in terms of shape function matrix N (the final integrals should be in terms of natural coordinates). *Do not* evaluate the integrals in the stiffness matrix. Writing the equations as $\begin{bmatrix} K_{11} & K_{12} & ... \\ K_{21} & K_{22} & .. \\ ... & & \\ K_{41} & K_{42} & ... \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$, find the force vector on the RHS, and the temperatures at the nodes in terms of K_{ij}.

3. The bottom of a beam of length $L = 1$, moment of inertia $I = 1$, Young modulus $E = 1$, cross sectional area $A = 1$ and density $\rho = 210$, travelling with uniform velocity $v_0 = 1$ is suddenly fixed at time $t = 0$ as shown in Fig. 3. Modelling the entire beam by one beam element, and using the energy-momentum conserving algorithm given by (30)

$$\frac{\hat{\boldsymbol{u}}_{n+1} - \hat{\boldsymbol{u}}_n}{t_\Delta} = \frac{\hat{\boldsymbol{v}}_n + \hat{\boldsymbol{v}}_{n+1}}{2},$$

$$\boldsymbol{M}\left(\frac{\hat{\boldsymbol{v}}_{n+1} - \hat{\boldsymbol{v}}_n}{t_\Delta}\right) + \boldsymbol{K}\left(\frac{\hat{\boldsymbol{u}}_n + \hat{\boldsymbol{u}}_{n+1}}{2}\right) = \frac{\hat{\boldsymbol{f}}_n + \hat{\boldsymbol{f}}_{n+1}}{2},$$

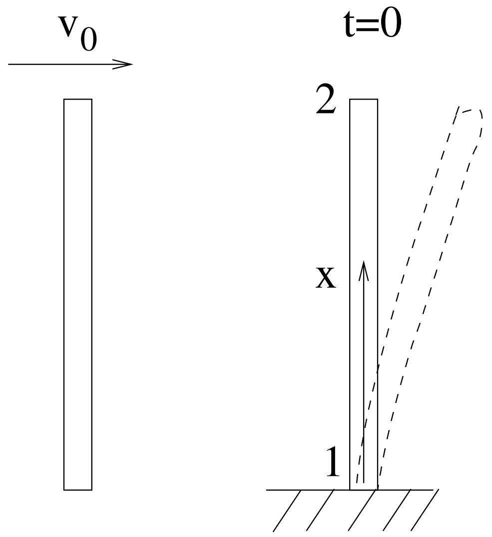

Figure 3: Problem 3

find the values of the displacement and velocity (in the transverse direction) of the top node of the beam at time $t_1 = t_\Delta = 1$. Are any of the three quantities, linear momentum, angular momentum and total energy (kinetic+ strain) conserved (both at a continuum and at an algorithmic level) in this problem? Give a mathematical justification (without actually computing any of these quantities).

Some Relevant Formulae

For a beam element of length L:

$$\boldsymbol{M}^{(e)} = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}, \quad \boldsymbol{K}^{(e)} = \begin{bmatrix} a & b & -a & b \\ b & 2c & -b & c \\ -a & -b & a & -b \\ b & c & -b & 2c \end{bmatrix},$$

where $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 2EI/L$.
Shape functions for a 4-node quadrilateral element:

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i), \quad i = 1,4.$$

Balance of linear and angular momenta, and the balance of energy (in the context of linear elasticity):

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \int_S \boldsymbol{t}\, dS + \int_V \rho \boldsymbol{b}\, dV,$$
$$\frac{d}{dt}\int_V \rho(\boldsymbol{X}\times\boldsymbol{v})\, dV = \int_S \boldsymbol{X}\times\boldsymbol{t}\, dS + \int_V \rho\boldsymbol{X}\times\boldsymbol{b}\, dV,$$
$$\frac{d}{dt}\int_V \rho\left[\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} + W(\boldsymbol{\epsilon})\right] dV = \int_S \boldsymbol{t}\cdot\boldsymbol{v}\, dS + \int_V \rho\boldsymbol{b}\cdot\boldsymbol{v}\, dV,$$

where $W(\boldsymbol{\epsilon}) = \boldsymbol{\epsilon} : \mathbb{C} : \boldsymbol{\epsilon}/2$ is the strain-energy density function.