Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require.

Date: 23/4/2012.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. We saw in the test that the governing equation and boundary condition for the torsion conjugate function are given by (20)

$$\boldsymbol{\nabla}^2 g \equiv \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0, \quad \text{in } A,$$
$$g = \frac{1}{2}(x^2 + y^2) \quad \text{on } C.$$

Let g_δ be the variation of g. Using this notation, formulate the variational statement for the above problem. Develop a finite element method based on this variational formulation in terms of natural coordinates. You may assume a quadrilateral element for specifying the limits of integration. If one nine-node element is used to discretize the domain shown in Fig. 1 with $a = 2$, $b = 4$, *describe* how you would find the value of g at the nine nodes. You can use

$$\begin{bmatrix} K_{11} & K_{12} & \dots \\ K_{21} & K_{22} & \dots \\ K_{31} & K_{32} & \dots \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix},$$

without actually computing the K_{ij}, to carry out the description.

2. Develop the axisymmetric variational, and subsequently the axisymmetric finite element formulation for the equation (35)

$$\boldsymbol{\nabla}^2 p + k^2 p = 0,$$

where p is the pressure, k is a given constant, and

$$\boldsymbol{\nabla}^2 p \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial^2 p}{\partial z^2}.$$

Assume that on part of the boundary Γ_p the pressure is prescribed, while on the remaining part Γ_h, $(\boldsymbol{\nabla} p \cdot \boldsymbol{n}) = \bar{h}$, where $\bar{h}$ is the prescribed normal pressure gradient, and $\boldsymbol{\nabla} p \equiv (\partial p/\partial r, \partial p/\partial z)$. As usual, the formulation should be in terms of natural coordinates, and you can use the integration limits corresponding to a quadrilateral element.

(a) What is the boundary condition to be imposed on the z-axis?

(b) An axisymmetric domain is discretized using a single Q4 axisymmetric element as shown in Fig. 2. By working with natural coordinates, find the 'consistent load vector' assuming $\bar{h} =$ constant.

3. For the torsion of a circular cylinder of length L which is fixed at one end and subjected to a time-varying moment $M(t)$ at the other, the displacement field is given by (45)

$$u_x = -y\phi(z,t); \quad u_y = x\phi(z,t); \quad u_z = 0,$$

where the z-axis is directed along the axis of the cylinder, and ϕ denotes the angular displacement.

(a) Using the stress-strain relation $\tau_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$, where $\epsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ and $\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$, find the expressions for the stresses τ_{xx}, τ_{xy} etc.

(b) Using the equations

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{\nabla} \cdot \boldsymbol{\tau},$$

find the governing equation for ϕ. Using the equation

$$\int_A (x\tau_{yz} - y\tau_{xz})\, dA = M(t),$$

where A represents the circular cross-section at the top, find the boundary condition for ϕ at $z = L$. The boundary condition will involve the polar moment of inertia J (*do not* evaluate the integral for J). The initial conditions are given by $\phi(z,0) = \phi_0(z)$, and $\dot{\phi}(z,0) = v_0(z)$.

(c) Multiply the governing equation for ϕ by $\dot{\phi} \equiv \partial\phi/\partial t$, and integrate with respect to z over the interval $[0, L]$. Assume ρ, λ and μ to be constant. By carrying out an appropriate integration by parts, derive an equation of the form

$$\frac{dH}{dt} = \text{RHS}, \tag{1}$$

where

$$H = \int_0^L [\ldots]\, dz, \tag{2}$$

and the RHS comprises of boundary terms involving $M(t)$. If $M(t)$ is suddenly set to zero, deduce a 'conservation law' for H from Eqn. (1).

(d) Multiply the governing equation for ϕ that you derived in (3b) by ϕ_δ, where ϕ_δ is the variation of ϕ, and again integrate with respect to z. By carrying out an appropriate integration by parts, derive the variational formulation for ϕ. By carrying out a spatial discretization of the form $\phi = \boldsymbol{N}\hat{\boldsymbol{\phi}}$, derive the semi-discrete form of the finite element equations.

(e) Propose a time-stepping strategy over the interval $[t_n, t_{n+1}]$, such that the 'finite-element' H obeys the same conservation law as the continuum conservation law that you derived in (3c) above. *Prove* this conservation result. If the initial conditions are $\phi_0(z) = cz$, $v_0(z) = 0$, where c is a constant, and subsequently, the circular bar is let go (while still being attached at the base), so that it undergoes torsional oscillations, will H given by Eqn. (2) be conserved? Justify mathematically.

(f) Using the governing equation for ϕ that you derived in (3b), find the continuum eigenfrequencies for the fixed-free configuration (fixed at $z = 0$ and free at $z = L$). Using the semi-discrete formulation that you derived in (3d), state the equation for finding the finite element eigenfrequencies. Taking $\rho = \mu = \lambda = 1$, $L = \pi/2$, and using one quadratic element, compare the first two finite element eigenfrequencies with the analytical ones (You may use $\pi^2 \approx 10$ if you like).

Some Relevant Formulae

For a quadratic 1-D element of length L with midnode at the center:

$$\int_0^L \boldsymbol{N}^T \boldsymbol{N}\, dz = \frac{L}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\int_0^L \frac{d\boldsymbol{N}^T}{dz} \frac{d\boldsymbol{N}}{dz}\, dz = \frac{1}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}.$$

Shape functions for a 4-node quadrilateral element:

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i), \quad i = 1, 4.$$

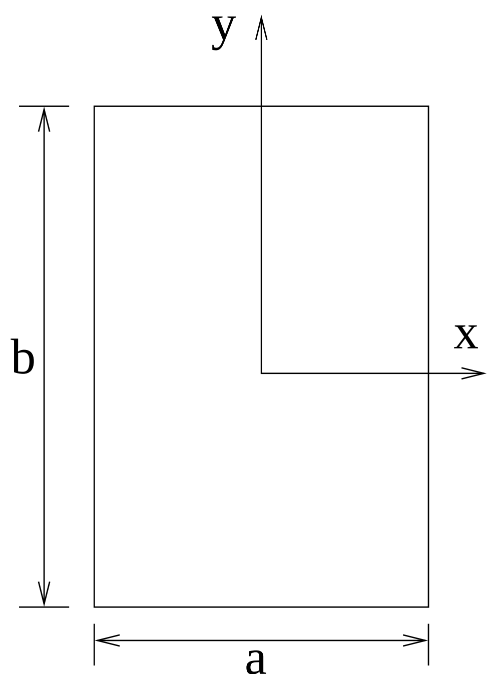

Figure 1: Problem 1

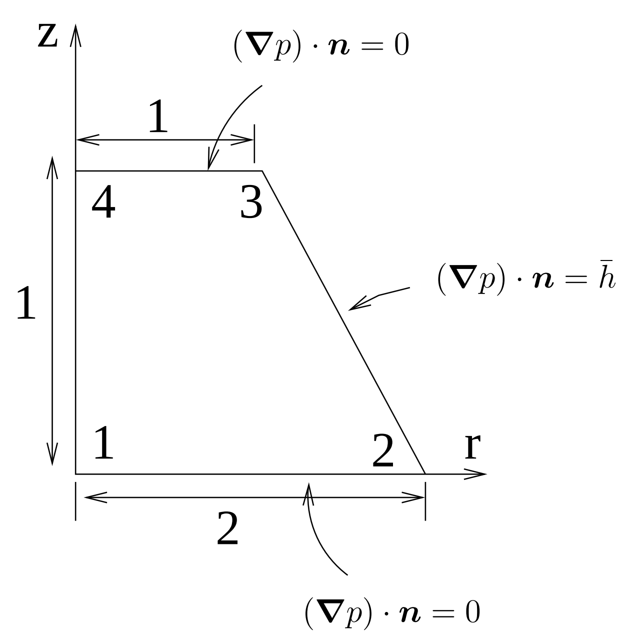

Figure 2: Problem 2