Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require.

Date: 22/4/2013.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. Using a two-element mesh, write the system of equations to be solved for finding the displacement at the point of loading P in the setup shown in Fig. 1. Write the equations in the form $\boldsymbol{K}\hat{\boldsymbol{u}} = \hat{\boldsymbol{f}}$, and then incorporate appropriate boundary conditions into it. You may directly use the given element matrices to construct the $\boldsymbol{K}$ matrix. *Do not attempt to solve these equations.* (20)

2. A tangential traction t_0 acts along edge 2-3-6 of a 9-node plane-stress element as shown in Fig. 2. The coordinates of the points 1 to 9 are $(0,0)$, $(4,0)$, $(2,2)$, $(0,2)$, $(2,0)$, $(3.5,1)$, $(1,2)$, $(0,1)$ and $(1.5,1)$. Find the consistent load vector for this element. Assume the thickness of the domain to be unity. (Hint: Express the tangent vector in terms of the parameter s that parametrizes the boundary.) (35)

3. The governing equation for the sphericallly symmetric vibrations of a sphere of radius a is given by (45)

$$\frac{1}{c^2}\left[\frac{\partial^2 u}{\partial t^2} - b(r,t)\right] = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)\right) \qquad (1)$$

where c is a given material constant, r is the radial coordinate in the spherical

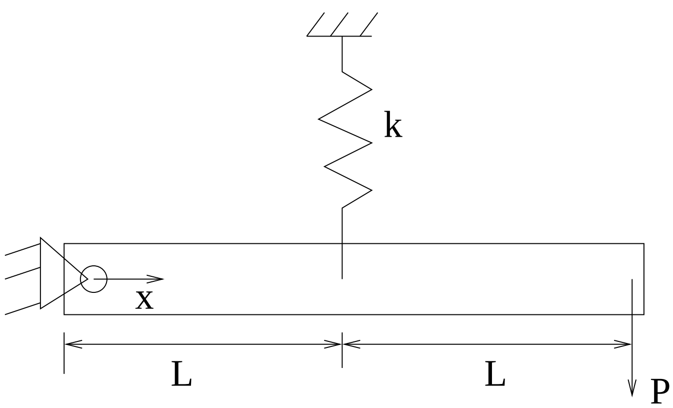

Figure 1: Problem 1

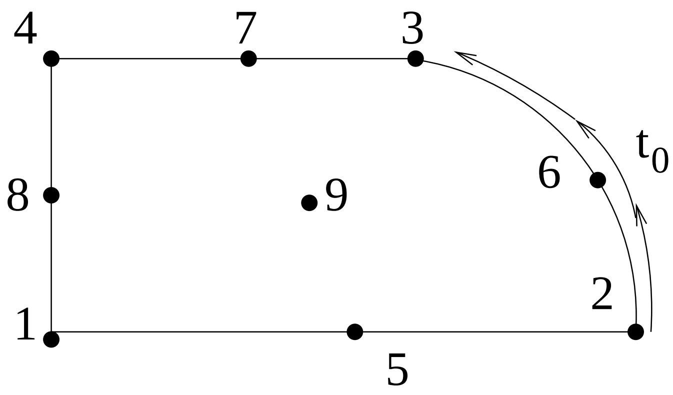

Figure 2: Problem 2

coordinate system, t denotes time, u is the radial displacement, and $b(r,t)$ is a given body force that is a function of r and t. Assume that $u|_{r=a} = 0$.

(a) By multiplying Eqn. (1) by the variation u_δ, and carrying out the integration over the *domain of the sphere*, find the variational formulation.

(b) By assuming $\phi_\delta = \partial u/\partial t$ in the above variational formulation, derive an equation of the form

$$\frac{dH}{dt} = \text{RHS}, \tag{2}$$

where

$$H = \int_0^a [\ldots]\, dr, \tag{3}$$

and the RHS comprises of terms involving $b(r,t)$. If $b(r,t)$ is suddenly set to zero, deduce a 'conservation law' for H from Eqn. (2).

(c) By carrying out a spatial discretization of the form $u = \boldsymbol{N}\hat{\boldsymbol{u}}$, derive the semi-discrete form of the finite element equations. Is the stiffness matrix $\boldsymbol{K}$ symmetric?

(d) Propose a time-stepping strategy over the interval $[t_n, t_{n+1}]$, such that the 'finite-element' H obeys the same conservation law as the continuum conservation law that you derived in (3b) above. *Prove* this conservation result.

(e) Setting b to zero in Eqn. (1), find the equation and state the boundary conditions for determining the continuum eigenfrequencies for the radial vibrations of the sphere. *Do not attempt to solve this equation.*

(f) Using the semi-discrete formulation that you derived in (3c), state the equation for finding the finite element eigenfrequencies. Using one quadratic element, find the first eigenfrequency. You may directly use the formulae at the back, and derive any formulae that are not listed there.

Some Relevant Formulae

For a linear bar element of length L:

$$\boldsymbol{K}^{(e)} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For a beam element of length L:

$$\boldsymbol{K}^{(e)} = \begin{bmatrix} a & b & -a & b \\ b & 2c & -b & c \\ -a & -b & a & -b \\ b & c & -b & 2c \end{bmatrix},$$

where $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 2EI/L$.

Volume of a sphere:

$$\int_0^a 4\pi r^2 \, dr = \frac{4}{3}\pi a^3.$$

For a quadratic 1-D element of length a with midnode at the center:

$$\int_0^a \boldsymbol{N}^T \boldsymbol{N} \, dr = \frac{a}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\int_0^a r^2 \boldsymbol{N}^T \boldsymbol{N} \, dr = \frac{a^3}{420}\begin{bmatrix} 2 & -4 & -5 \\ -4 & 64 & 24 \\ -5 & 24 & 44 \end{bmatrix},$$

$$\int_0^a \frac{d\boldsymbol{N}^T}{dr}\frac{d\boldsymbol{N}}{dr} \, dr = \frac{1}{3a}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix},$$

$$\int_0^a \frac{1}{r^2}\frac{d(r^2\boldsymbol{N}^T)}{dr}\frac{d(r^2\boldsymbol{N})}{dr} \, dr = \frac{a}{15}\begin{bmatrix} 7 & -4 & 2 \\ -4 & 48 & -24 \\ 2 & -24 & 57 \end{bmatrix}.$$