Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require.

Date: 22/4/2014.
Duration: 2.00 p.m.–5.00 p.m.
Maximum Marks: 100

1. Derive the variational formulation for the governing equation and boundary condition given by (35)

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = k^2 \boldsymbol{u}, \quad \text{in } \mathit{\Omega},$$
$$(\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{n} = \bar{\boldsymbol{h}} \quad \text{on } \mathit{\Gamma},$$

where $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is expressed as

$$w_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

You may directly use the relation $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. From this point on consider the 2D version of the above problem, i.e., assume $\boldsymbol{u} = (u_x(x,y), u_y(x,y), 0)$, $\bar{\boldsymbol{h}} = (\bar{h}_x, \bar{h}_y, 0)$; take the thickness to be unity. Consider a three-node triangle with coordinates of nodes 1, 2 and 3 given by $(a, 0)$, $(0, b)$ and $(0, 0)$. Find the '$\boldsymbol{B}$' matrix that links gradients of $\boldsymbol{u}$ to the nodal degrees of freedom $\hat{\boldsymbol{u}}$ in terms of natural coordinates for this triangle. Assuming that edge 1-2 is lying on the boundary, where $\bar{\boldsymbol{h}} = (x, y, 0)$, find the consistent load vector.

2. We wish to develop the finite element formulation for the torsion of circular cylinders of variable diameter (see Fig. 1) in the r-z plane using two-dimensional elements (as in an axisymmetric formulation). The lateral surface is traction free, and the torque T is generated by prescribed tractions $t_\theta = t_\theta(r)$ on the top surface. We assume $u_\theta(r, z)$ to be the only nonzero component, and hence the only nonzero strains and stresses are (30)

$$\epsilon_{r\theta} = \frac{r}{2} \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right), \qquad \tau_{r\theta} = 2G\epsilon_{r\theta},$$
$$\epsilon_{\theta z} = \frac{1}{2} \frac{\partial u_\theta}{\partial z}, \qquad \tau_{\theta z} = 2G\epsilon_{\theta z}.$$

The equilibrium equations reduce to $(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r = (\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z = 0$, and

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta = \frac{\partial \tau_{\theta r}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0,$$

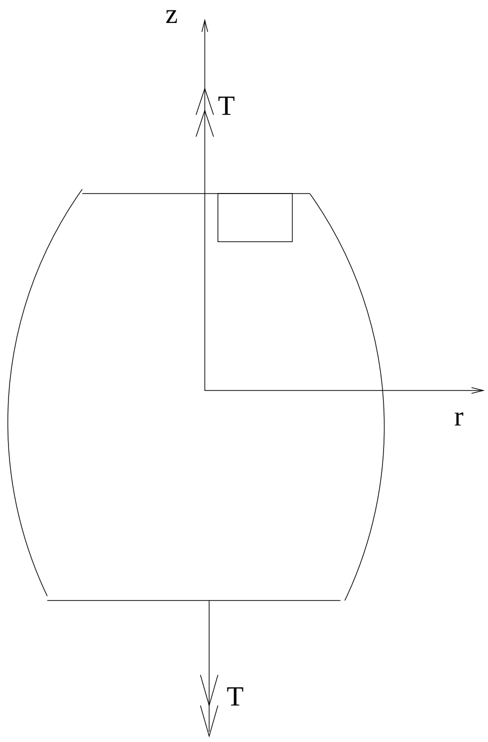

Figure 1: Problem 2

Develop the variational formulation on the r-z plane, and the associated finite element formulation for determining u_θ. *Formulate such that the* $\boldsymbol{K}$ *matrix is symmetric* (Hint: Recall the axisymmetric formulation covered in the class). The $\boldsymbol{K}$ matrix should be in expressed in terms of $\boldsymbol{B}$ as $\int_{-1}^{1}\int_{-1}^{1} \ldots \, d\xi \, d\eta$ (*do not* evaluate this integral), while $\boldsymbol{B}$ should be expressed in terms of the derivatives of the shape functions with respect to r and z, say, for a 4-node quadrilateral element (*do not* find these derivatives in terms of ξ and η). Also find the consistent load vector corresponding to $t(\theta) = cr$, on the top edge of the rectangular element shown whose coordinates are (r_1, z_1) (bottom left node), (r_2, z_1) (bottom right node), (r_2, z_2) and (r_1, z_2), where c is a constant in the form of an integral $\int_{-1}^{1} \ldots \, d\xi$, where the integrand should be a function of ξ (*do not* evaluate this integral).

3. The scalar wave equation is given by (35)

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \boldsymbol{\nabla}^2 p,$$

where c is a constant (wave speed).

(a) Assuming p_δ to be the variation of p, develop the variational formulation.

(b) By taking $p_\delta = \partial p/\partial t$ in the variational formulation, derive a relation of the form

$$\frac{dH}{dt} = \text{RHS},$$

where

$$H = \int_{\Omega} [\ldots] \, d\Omega,$$

and the RHS comprises of forcing boundary terms involving $\boldsymbol{\nabla} p$. If this forcing is suddenly set to zero, deduce a conservation law for H.

(c) By carrying out a spatial discretization of the form $p = \boldsymbol{N}\hat{p}$, derive the semi-discrete form of the finite element equations.

(d) Propose a time-stepping strategy over the interval $[t_n, t_{n+1}]$, such that the 'finite-element' H obeys the same conservation law as the continuum conservation law that you derived in (3b) above in the absence of forcing terms. *Prove* this conservation result.

(e) From now on, consider only wave propagation along the x-direction in a duct with unit area, i.e., $p = p(x,t)$, and $\boldsymbol{\nabla}^2 p = \partial^2 p/\partial x^2$. If we consider wave propagation in a one-dimensional duct of length L, where $p =$ constant on the left end $x = 0$ and $\partial p/\partial x = 0$ on the right end $x = L$, is the quantity H conserved in the continuum problem?

(f) If $p = 0$ at $x = 0$ and $\partial p/\partial x = \sin \omega t$ at $x = L$, find the continuum 'periodic steady state' solution (i.e., the solution attained after the initial transients have died out). Taking $\omega = 1$, $c = 1$, $L = \pi$, find the spatial part of the solution using one quadratic element, and compare the nodal solution against the analytical one (Take π^2 to be 10 if you don't have a calculator).

Some Relevant Formulae

For a quadratic 1-D element of length L with midnode at the center:

$$\int_0^L \boldsymbol{N}^T \boldsymbol{N} \, dx = \frac{L}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\int_0^L \frac{d\boldsymbol{N}^T}{dx} \frac{d\boldsymbol{N}}{dx} \, dx = \frac{1}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix},$$

$$\int_0^L \boldsymbol{N}^T \, dx = \frac{L}{6} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

Shape functions for a 4-node quadrilateral element:

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i), \quad i = 1, 4.$$