Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end. Derive any other formulae that you may require.

Date: 21/4/2015.
Duration: 2.00 p.m.–5.00 p.m.
Maximum Marks: 100

1. For a certain problem involving a vector-valued function $\boldsymbol{u} = (u_x, u_y)$, we get a term in the variational formulation as (25)

$$\int_\Omega \boldsymbol{\nabla u}_\delta : \boldsymbol{\nabla u}\, d\Omega,$$

where the domain may be considered as two-dimensional with independent coordinates (x, y). The stiffness matrix associated with the above term can be written as $\int_\Omega \boldsymbol{B}^T \boldsymbol{B}\, d\Omega$. Describe using equations how you would find the $\boldsymbol{B}$ matrix using natural coordinates (ξ, η) for a 3-node triangular element having 6 degrees of freedom ((u_x, u_y) degrees of freedom at each node), and coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). Assuming a unit thickness for the two-dimensional domain, also find $d\Omega$ in terms of (ξ, η).

2. This problem is *unconnected* to the previous problem (although it may appear to be), and can be solved independently. The governing equation for the velocity field $\boldsymbol{v}$ in transient Stokes flow with a prescribed pressure gradient $\boldsymbol{G}(t)$ is (35)

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{G}(t) + \mu \boldsymbol{\nabla}^2 \boldsymbol{v},$$

   (a) Develop the variational formulation for the above governing equation. You may leave the boundary term as it is at this stage.

   (b) Consider an axisymmetric velocity field $(0, 0, v_z(r, t))$ with prescribed pressure gradient $\boldsymbol{G}(t) = (0, 0, g(t))$ in a circular tube of radius a with no-slip boundary conditions at the wall $r = a$. Develop the semi-discrete formulation for finding $v_z(r, t)$.

   (c) Describe a time-stepping strategy to solve the semi-discrete formulation that you have developed. You should use an unconditionally stable scheme (justify why it is so, although you need not prove this).

   (d) If $g(t) = t$ and the initial velocity v_z is $(a^2 - r^2)$, state the matrix equation (with boundary conditions incorporated) for finding the nodal velocities at time $t_1 = t_\Delta$. *You need not simplify this matrix equation.* You may use one quadratic element, and directly use the expressions at the back.

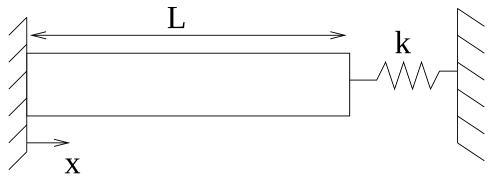

Figure 1: Problem 4

3. Derive the '1D' governing equation for the longitudinal vibration of a bar with Young modulus E, cross-sectional area A and length L, fixed at one end and spring-supported at the other (see Fig. 1) using the 1D approximations $\epsilon = \partial u/\partial x$, $\tau = E\epsilon$ etc., and neglecting body forces in the general elastodynamics equation $\boldsymbol{\nabla} \cdot \boldsymbol{\tau} = \rho \partial^2 \boldsymbol{u}/\partial t^2$. The governing equation should be in terms of u. The boundary conditions are (40)

$$u\big|_{x=0} = 0, \quad (A\tau + ku)\big|_{x=L} = 0.$$

   (a) The bar is given an initial displacement $u(0) = \delta x/L$ (so that the spring gets compressed by an amount δ) and then released. Taking the initial velocity to be zero, find the total energy (including the potential energy in the spring) in the system at $t = 0$. Take $d\Omega = A\,dx$.

   (b) Develop the variational formulation for the 1D governing equation that you have derived.

   (c) By making a particular choice of the variation u_δ in the variational formulation (Hint: Choose it to be either u or some derivative of u with respect to t), determine if the total energy of the continuum system is conserved.

   (d) Develop the semi-discrete formulation for the vibrations of the structure.

   (e) In case the (continuum) energy is conserved, develop a time-stepping strategy that mimics this energy-conserving nature of the continuum (if the total energy is not conserved, then justify why it is not conserved), and prove that your strategy conserves energy. Using one linear finite element (you may directly use the matrices at the back), find the total (finite element) energy at $t = 0$. Is this energy the same as the total (continuum) energy that you found in part (a)?

   (f) Using one linear element for the bar, find the response at time $t_1 = t_\Delta$ using your proposed strategy, i.e., find the nodal displacement at $x = L$ and $t_1 = t_\Delta$.

Some Relevant Formulae

For a linear bar element of length L:

$$\boldsymbol{K}^{(e)} = d \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \boldsymbol{M}^{(e)} = h \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

where $d = EA/L$, $h = \rho AL/6$.
For a quadratic 1-D element of length a with midnode at the center:

$$\int_0^a \boldsymbol{N}^T \boldsymbol{N}\, dr = \frac{a}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \qquad \int_0^a \boldsymbol{N}^T \boldsymbol{N} r\, dr = \frac{a^2}{60}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 16 & 4 \\ -1 & 4 & 7 \end{bmatrix},$$

$$\int_0^a \frac{d\boldsymbol{N}^T}{dr}\frac{d\boldsymbol{N}}{dr}\, dr = \frac{1}{3a}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}, \quad \int_0^a \frac{d\boldsymbol{N}^T}{dr}\frac{d\boldsymbol{N}}{dr} r\, dr = \frac{1}{6}\begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix},$$

$$\int_0^a \boldsymbol{N}^T\, dr = \frac{a}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \qquad \int_0^a \boldsymbol{N}^T r\, dr = \frac{a^2}{6}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

The gradient of a vector field $\boldsymbol{v}$ in a cylindrical (r, θ, z) system is

$$[\boldsymbol{\nabla v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}.$$