Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given in each part. Derive any other formulae that you may require.

Date: 21/4/2016.
Duration: 2.00 p.m.–5.00 p.m.
Maximum Marks: 100

1. Derive the variational formulation for the governing equation and boundary condition given by (30)

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = k^2 \boldsymbol{u}, \quad \text{in } \varOmega,$$
$$(\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{n} = \bar{\boldsymbol{h}} \quad \text{on } \varGamma,$$

where $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is expressed as

$$w_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

You may directly use the relation $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. From this point on, consider the axisymmetric version of this problem, i.e,, $u_r = u_z = 0$, and $u_\theta(r, z)$ is the only nonzero component. Since there is no dependence of any of the variables on θ, the problem is axisymmetric, and can be solved on the r-z plane. You may directly use

$$(\boldsymbol{\nabla} \times \boldsymbol{u})_r = -\frac{\partial u_\theta}{\partial z},$$
$$(\boldsymbol{\nabla} \times \boldsymbol{u})_z = \frac{1}{r} \frac{\partial (r u_\theta)}{\partial r}.$$

Consider a four-noded rectangular element in the r-z plane with coordinates of its nodes 1 to 4 given by $(0, 0)$, $(2a, 0)$, $(2a, 2b)$ and $(0, 2b)$. Thus, we have $r = (1+\xi)a$ and $z = (1+\eta)b$. Let V represent the three-dimensional domain.

(a) If the 'stiffness matrix' is given by $\int_V \boldsymbol{B}^T \boldsymbol{B} \, dV$, then find an expression for $\boldsymbol{B}$ in terms of N_i, $\partial N_i/\partial \xi$, $\partial N_i/\partial \eta$, $i = 1, \ldots, 4$. You need not write the shape functions N_i. Note that the derivatives of N_i should be with respect to the natural coordinates and not with respect to (r, z). Next, express dV in terms of $d\xi \, d\eta$.

(b) If the mass matrix (the term associated with k^2) is expressed as $\int_V \boldsymbol{N}^T \boldsymbol{N}$, then state what $\boldsymbol{N}$ is in terms of the N_i, $i = 1, \ldots, 4$.

(c) If $\bar{h}_\theta = \gamma z$, where γ is a given constant, acts on the edge 2-3 of the above element, find the corresponding consistent load vector.

2. If $u_r = u_r(r)$ and $u_\theta = u_z = 0$, the governing equation for a linear elastic structure is given by (35)

$$\frac{1}{r}\frac{d(r\tau_{rr})}{dr} - \frac{\tau_{\theta\theta}}{r} = 0. \tag{1}$$

(a) Take the length of the cylinder along z to be unity. Using the relations

$$\epsilon_{rr} = \frac{du_r}{dr}; \quad \epsilon_{\theta\theta} = \frac{u_r}{r}, \quad \text{(other strains zero)}$$
$$\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon}) + 2\mu\boldsymbol{\epsilon},$$

derive the variational formulation corresponding to Eqn. (1). One of the integrands in this variational formulation should be expressed in the form $\boldsymbol{\epsilon}_c^T(\boldsymbol{u}_\delta)\boldsymbol{\tau}_c$, where

$$\boldsymbol{\tau}_c = \begin{bmatrix} \tau_{rr} \\ \tau_{\theta\theta} \end{bmatrix} = \boldsymbol{C}\begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \end{bmatrix} = \boldsymbol{C}\boldsymbol{\epsilon}_c,$$

and where $\boldsymbol{C}$ is a matrix that you have to find in terms of λ and μ.

(b) For a two-node linear element with radial coordinates r_1 and r_2 , write the element stiffness matrix in terms of the strain-displacement matrix $\boldsymbol{B}$, which in turn should be in terms of the shape functions N_1, N_2, and the derivatives $dN_1/d\xi$ and $dN_2/d\xi$. *Do not evaluate any integrals that arise.*

(c) Use the above formulation to solve the following problem. A hollow cylinder of inner radius $a-\Delta$ and outer radius b is shrink-fitted around a solid cylinder of radius a as shown in Fig. 1 (the individual cylinders *before* shrink fitting are shown in the upper part of the figure, and the assembly *after* shrink fitting is shown in the lower part). Both cylinders are made of the same material. Plane strain conditions are maintained during the shrink fitting process by constraining the motion along the z-direction (but *not* along the other directions) as shown in the figure, i.e., $u_z = \epsilon_{zz} = 0$. Assume the process to be axisymmetric. Since Δ is assumed to be small in relation to a, the displacements at $r = a$ for the inner and outer cylinders are related as $\left[u_r^{(2)} - u_r^{(1)}\right]_{r=a} = \Delta$, where $u_r^{(1)}$ and $u_r^{(2)}$ are the displacement fields in the inner and outer cylinders. If we use one linear element to model the inside cylinder and one linear element to model the outside cylinder, then for some values of a, b, λ and μ, the element stiffness matrices for the inner and outer regions are given by

$$\boldsymbol{K}^{(1)} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}, \quad \boldsymbol{K}^{(2)} = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix},$$

If $\Delta = 0.03$, find the displacement field in the two cylinders as a function of the natural coordinate ξ in each of the two cylinders, i.e., $u_r^{(1)} = u_r^{(1)}(\xi)$ and $u_r^{(2)} = u_r^{(2)}(\xi)$

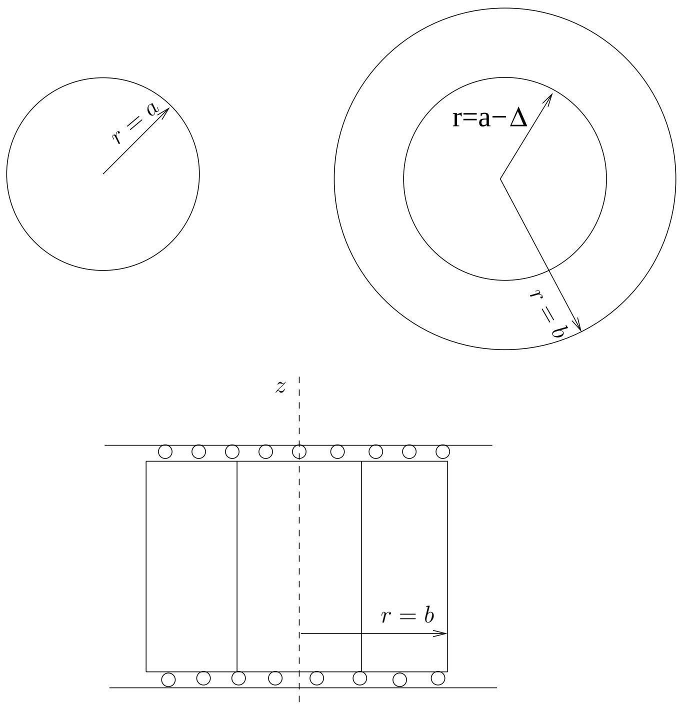

Figure 1: Problem 2.

3. We are interested in finding what quantities are conserved in linear elastodynamics in the presence of a temperature change ΔT. Let the usual governing equations hold except that now $\boldsymbol{\tau}$ is given by (35)

$$\boldsymbol{\tau} = \mathbf{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0),$$
$$\boldsymbol{\tau}_c = \boldsymbol{C}(\boldsymbol{\epsilon}_c - (\boldsymbol{\epsilon}_0)_c)$$

where $\boldsymbol{\epsilon}_0 = \alpha \Delta T \boldsymbol{I}$, $(\boldsymbol{\epsilon}_0)_c = (\alpha\Delta T, \alpha\Delta T, \alpha\Delta T, 0, 0, 0)^T$, $\mathbf{C}$ is the constitutive matrix which is assumed to be symmetric and positive definite ($\mathbf{C}_{ijkl} = \mathbf{C}_{klij}$), and $\boldsymbol{C}$ is the engineering form (6×6 matrix) of $\mathbf{C}$. Assume that there is no damping.

   (a) Suppose that there is some mechanical loading in the form of tractions or body forces, and thermal loading due to a temperature difference with respect to the ambient $\Delta T(\boldsymbol{x}, t)$. At some instant, the body forces and tractions on the entire surface of the body are set to zero, and ΔT is set to a constant with respect to time, i.e., the temperature difference with respect to the ambient is $\Delta T(\boldsymbol{x})$. Deduce whether, from this moment on, the linear and angular momenta are conserved in the continuum setting. Also deduce whether the sum of kinetic and (mechanical) strain energy is conserved. If not, deduce a 'modified energy' which is conserved.

   (b) Develop a time-stepping scheme that mimics the conservation properties of the continuum that you have derived in the part above. You may directly use the semi-discrete form

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f},$$

where

$$
\begin{aligned}
\boldsymbol{M} &= \int_V \rho \boldsymbol{N}^T \boldsymbol{N} \, dV, \\
\boldsymbol{K} &= \int_V \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \, dV, \\
\boldsymbol{f} &= \int_{S_t} \boldsymbol{N}^T \bar{\boldsymbol{t}} \, dS + \int_V \rho \boldsymbol{b} \, dV + \int_V \boldsymbol{B}^T \boldsymbol{C} (\boldsymbol{\epsilon}_0)_c \, dV.
\end{aligned}
$$

Prove the conservation properties of the time-stepping scheme that you are proposing.