Indian Institute of Science, Bangalore

ME 257: Endsemester Exam

Note: Some relevant formulae are given at the end which you can directly use. Derive any other formulae that you may require.

Date: 21/4/2017.
Duration: 9.00 p.m.–12.00 noon
Maximum Marks: 100

1. The structure shown in Fig. 1 is subjected to a vertical force F at the center of the horizontal member as shown. All members have the same Young modulus E, a square cross section of area A and moment of inertia I, and are of length L. Directly using the given finite element matrices *and symmetry*, find the system of equations to find the deflection under the load F. Express these equations in matrix form. Do not attempt to solve these equations. Assume that the horizontal member is simply placed on the vertical bars without welding them together. (25)

2. The homogeneous radially symmetric transient heat conduction equation for the temperature field $T(r,t)$ on a spherical domain of radius a is given by (40)

$$\frac{1}{\beta}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right), \tag{1}$$

where β is a constant, and r denotes the (spherical) radial coordinate.

   (a) Assuming the initial temperature to be $f(r)$ and the boundary condition to be $T|_{r=a} = 0$, find the analytical solution that meets the boundary condition, governing equation and initial condition. In order to derive this solution, assume the temperature to be of the form $g(r)h(t)/r$. You

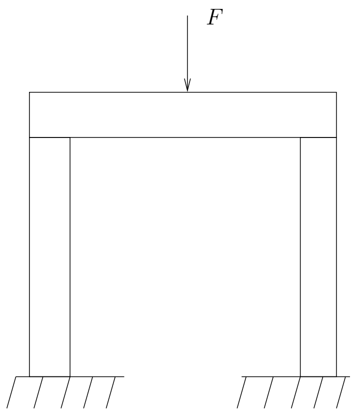

Figure 1: Structure subjected to vertical force F.

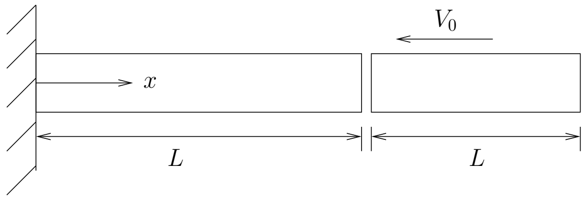

Figure 2: Elastic bar of length L impacting a stationary elastic bar of length L constrained by a rigid wall.

may make an 'intelligent guess' for $h(t)$ (Hint: The temperature at the center $r = 0$ is finite).

(b) Develop the variational formulation corresponding to Eqn. (1) subject to the boundary conditions in part (a).

(c) Discretize this variational formulation in the spatial variables using one quadratic element to develop the semi-discrete formulation.

(d) Discretize in time using the backward Euler scheme ($\alpha = 1$) to develop the fully discrete finite element formulation.

(e) Given that the $\Delta t = \beta = a = 1$, $f(r) = 1 - r^2$, and using one quadratic element, write the matrix equations for finding the temperature at $r = 0$ and $r = 1/2$ after the first time step, i.e., at time $\Delta t = 1$. You need not solve the matrix equations.

3. Consider a bar of length L impacting another bar of length L whose end is constrained by a rigid wall as shown in Fig. 2. (35)

(a) Assume that the velocity of the combined bar just after contact occurs is zero for the region $0 \leq x < L$, and V_0 for the region $L \leq x \leq 2L$. Starting from the semi-discrete form

$$\boldsymbol{M}\ddot{\hat{\boldsymbol{u}}} + \boldsymbol{K}\hat{\boldsymbol{u}} = \boldsymbol{f},$$

where

$$\boldsymbol{M} = \int_{\Omega} \rho \boldsymbol{N}^T \boldsymbol{N}\, d\Omega,$$
$$\boldsymbol{K} = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B}\, d\Omega,$$
$$\boldsymbol{f} = \int_{\Omega} \rho \boldsymbol{N}^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}^T \bar{\boldsymbol{t}},$$

devise an energy-momentum conserving time-stepping strategy to find the evolution of the displacement field after contact takes place and assuming that the contact is maintained (i.e., the bars do not separate). You need not prove the conservation properties of your proposed algorithm in this part, but you may have to prove them in the next.

(b) After contact takes place and the bars start deforming, is linear momentum conserved in this problem? Is the total energy conserved? Explain. Show that your numerical strategy also conserves the conserved continuum quantity (If both are not conserved, you need not show anything).

(c) Approximating the two bars using a '1D' approximation, and using a total of two linear elements find the displacements at the nodes at time Δt assuming $\rho = A = L = V_0 = \Delta t = 1$. Since at $t = 0$ just after contact occurs, there is a discontinuity in the velocity field, evaluate $\boldsymbol{M}\hat{\boldsymbol{v}}_0$ as $\int_\Omega \rho \boldsymbol{N}^T \boldsymbol{v}_0 \, d\Omega$.

Some Relevant Formulae

For a linear bar element of length L:

$$\boldsymbol{K}^{(e)} = d \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \boldsymbol{M}^{(e)} = \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

where $d = EA/L$, $\alpha = \rho AL/6$.
For a quadratic 1-D element with midnode at the center:

$$\int_{-1}^{1} (1+\xi)^2 \boldsymbol{N}^T \boldsymbol{N} \, d\xi = \frac{1}{105} \begin{bmatrix} 4 & -8 & -10 \\ -8 & 128 & 48 \\ -10 & 48 & 88 \end{bmatrix},$$

$$\int_{-1}^{1} (1+\xi)^2 \frac{d\boldsymbol{N}^T}{d\xi} \frac{d\boldsymbol{N}}{d\xi} \, d\xi = \frac{1}{15} \begin{bmatrix} 6 & -12 & 6 \\ -12 & 64 & -52 \\ 6 & -52 & 46 \end{bmatrix},$$

$$\int_{-1}^{1} (1+\xi)^2 \boldsymbol{N}^T \, d\xi = \frac{1}{15} \begin{bmatrix} -2 \\ 24 \\ 18 \end{bmatrix}$$

For a beam element of length L:

$$\boldsymbol{K} = \phi \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix},$$

where $\phi = EI/L^3$.
For n an integer,

$$\int_0^a \sin^2 \frac{n\pi r}{a} \, dr = \int_0^a \cos^2 \frac{n\pi r}{a} \, dr = \frac{a}{2},$$

$$\int_0^a \sin \frac{n\pi r}{a} \sin \frac{m\pi r}{a} \, dr = \int_0^a \cos \frac{n\pi r}{a} \cos \frac{m\pi r}{a} \, dr = 0, \quad (m \neq n).$$