Indian Institute of Science, Bangalore

ME 257: Midsemester Test

Date: 26/2/2024.
Duration: 5.00 p.m.–6.30 p.m.
Maximum Marks: 100

1. Let V denote a spherical domain of radius a, and let S denote its surface. Let (r, θ, ϕ) denote spherical coordinates. In the following problem, we assume that the problem is axisymmetric, i.e., p is a function of (r, θ). A functional is defined as (55)

$$\Pi = \frac{1}{2} \int_V \left[\left(\frac{\partial p}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial p}{\partial \theta} \right)^2 - p^2 \right] dV - \int_S g(\theta) p \, dS,$$

where $dV = 2\pi r^2 \sin\theta \, dr \, d\theta$, and $dS = 2\pi a^2 \sin\theta \, d\theta$.

   (a) By setting the first variation of Π to zero, derive the variational formulation.

   (b) Using the variational formulation, derive the governing equation and boundary conditions that have been imposed in this problem.

2. A disc of inner radius a, outer radius b, and unit width along the z-direction is fixed rigidly at the inner boundary $r = a$, and on the outer boundary $r = b$ is subjected to a *constant* tangential traction s_0 as shown in Fig. 1. Body forces are zero. You may treat this as a two dimensional problem in terms of the polar coordinates (r, θ). (45)

   (a) It is given that $u_r = c_1 \cos\theta + c_2 \sin\theta$. Using the given boundary conditions, determine the constants c_1 and c_2.

   (b) By making appropriate assumptions based on the geometry and the nature of the loading, simplify the following equations

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{r\theta} = \frac{1}{2} \left[\frac{1}{r} \frac{\partial u_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) \right],$$

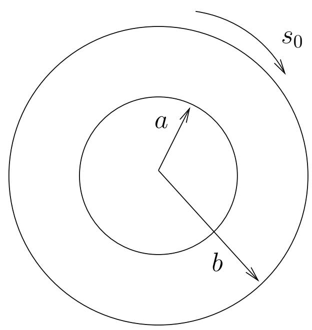

Figure 1: Hollow disk fixed rigidly at the inner boundary $r = a$, and subjected to a constant tangential traction s_0 on the outer boundary $r = b$.

$$\epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \qquad \operatorname{tr}\boldsymbol{\epsilon} = \epsilon_{rr} + \epsilon_{\theta\theta},$$
$$\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon},$$

where the Lame parameters (λ, μ) are constant.

(c) Let V denote the domain, and S_t denote the part of the surface on which tractions are applied. Specialize the variational formulation

$$\int_V \boldsymbol{\epsilon}(\boldsymbol{v}) : \boldsymbol{\tau}\, dV = \int_{S_t} \boldsymbol{v} \cdot \bar{\boldsymbol{t}}\, dS \quad \forall \boldsymbol{v}, \tag{1}$$

where $dV = r\, dr\, d\theta$ and $dS = a\, d\theta$ and $b\, d\theta$ for the inner and outer surfaces, respectively, to the problem at hand, i.e., write this equation in terms of the individual stress, strain, traction components etc. in the (r, θ) coordinate system.

(d) Discretize u_θ using the two-node element shape functions. Derive the strain-displacement matrix $\boldsymbol{B}$, and state the element stiffness matrix $\boldsymbol{K}$, and the element load vector in terms of $\boldsymbol{B}$, and the shape function matrix $\boldsymbol{N}$. *Do not* carry out the integrations in the expression for $\boldsymbol{K}$.

(e) Assuming that the components of $\boldsymbol{K}$ are K_{11}, K_{12} etc., and using a single 2-node element, determine the unknown nodal displacements in terms of these components after incorporating the appropriate boundary conditions.

Some Relevant Formulae

For axisymmetric problems in spherical coordinates

$$\boldsymbol{\nabla}\Phi = \frac{\partial \Phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\boldsymbol{e}_\theta,$$
$$\boldsymbol{\nabla}^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right).$$