Indian Institute of Science, Bangalore

ME 257: Midsemester Test

Date: 16/2/2025.
Duration: 9.30 a.m.–11.30 a.m.
Maximum Marks: 100

1. Let the governing equation on a domain Ω be given by (50)

$$\boldsymbol{\nabla} \times \boldsymbol{T} + \boldsymbol{F} = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{T}$ and $\boldsymbol{F}$ are second-order tensors, and

$$(\boldsymbol{\nabla} \times \boldsymbol{T})_{ij} = \epsilon_{irs} \frac{\partial T_{js}}{\partial x_r}.$$

Let the constitutive relation be given by

$$\boldsymbol{T} = E \boldsymbol{\nabla} \times \boldsymbol{\epsilon}, \tag{2}$$

where E is a constant, and $\boldsymbol{\epsilon}$ is a symmetric second-order tensor. By using a symmetric second-order tensor as a weighing function, and using indicial notation, develop the variational formulation corresponding to Eqn. (1), and derive the admissible kinematic and natural boundary conditions on the boundary Γ of the domain Ω. In your final expression, the integrals on Ω should be converted from indicial to tensorial notation, but the integrals on Γ *can be in indicial notation.* Thus, the natural boundary condition can also be expressed in terms of indicial notation without converting to tensorial notation (Hint: The bilinear term on Ω in your variational formulation may or may not be symmetric).

2. Assume the body forces to be zero in this problem. A thick-walled hollow sphere of inner radius a and outer radius b is subjected to a uniform pressure load p at its inner surface $r = a$ as shown in Fig. 1. The outer surface $r = b$ is in contact with a smooth rigid surface so that it cannot move radially outward. Our goal is to find the displacement and stress distribution in the sphere as a function of the spherical coordinates r-θ-ϕ where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. (50)

   (a) Make suitable assumptions about the nature of the displacement field taking into account the symmetry of the structure, loading etc. (e.g., $u_\phi = 0$; in this step *do not* attempt to find the exact solution since we want to find the solution using FEM), simplify the following strain-displacement relations:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right],$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \qquad \epsilon_{\theta\phi} = \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right)\right],$$

$$\epsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r}, \quad \epsilon_{r\phi} = \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right].$$

(b) With λ, μ denoting the Lame parameters, and with

$$\boldsymbol{C}_1 = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda \\ \lambda & \lambda+2\mu & \lambda \\ \lambda & \lambda & \lambda+2\mu \end{bmatrix}; \quad \boldsymbol{C}_2 = \mu \boldsymbol{I}_{3\times 3},$$

$$\boldsymbol{\tau}_c = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{C}_2 \end{bmatrix} \boldsymbol{\epsilon}_c,$$

specialize the variational formulation that we developed in class

$$\int_\Omega [\boldsymbol{\epsilon}_c(\boldsymbol{v})]^T \boldsymbol{\tau}_c \, d\Omega = \int_{\Gamma_t} \boldsymbol{v} \cdot \bar{\boldsymbol{t}} \, d\Gamma \quad \forall \boldsymbol{v}, \tag{3}$$

to the problem at hand, i.e., write this equation in terms of your nonzero displacement and stress components. You may take $d\Omega = 2\pi r^2 \sin\theta \, dr \, d\theta$, and $d\Gamma = 2\pi R^2 \sin\theta d\theta$ for the surface of a sphere of radius R.

(c) Discretize the nonzero displacement components in your formulation using one quadratic element with midnode at the center of the element.

(d) Derive the strain-displacement matrix $\boldsymbol{B}$ for this element.

(e) State the element stiffness matrix $\boldsymbol{K}$, and the element load vector $\boldsymbol{f}$ in terms of $\boldsymbol{B}$ and the shape function matrix $\boldsymbol{N}$. *Do not* carry out the integrations in the expressions for $\boldsymbol{K}$, but state them with the proper integration limits in the natural coordinate system. Find an explicit expression for $\boldsymbol{f}$ which includes unknown reactions if any.

(f) Writing your global set of equations $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$ with the elements of $\boldsymbol{K}$ denoted by K_{11}, K_{12} etc., and the elements of $\boldsymbol{f}$ as f_1, f_2 etc., show how you incorporate the appropriate boundary conditions for this problem into these set of equations.

Some Relevant Formulae

For a quadratic 1-D element of length L with midnode at the center:

$$N_1(\xi) = \frac{\xi(\xi-1)}{2}, \; N_2(\xi) = (1-\xi^2), \; N_3(\xi) = \frac{\xi(\xi+1)}{2}, \; dx = \frac{L}{2} d\xi,$$

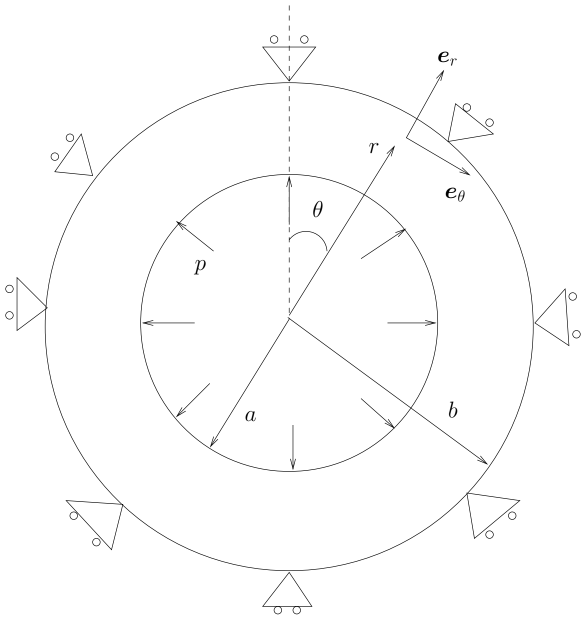

Figure 1: Problem 2.