Indian Institute of Science, Bangalore

ME 257: Midsemester Test

Date: 15/2/2026.
Duration: 10.00 a.m.–12.00 noon
Maximum Marks: 100

1. Let Ω denote a two-dimensional domain, and let $\boldsymbol{u} = (u_x, u_y)$ be a vector-valued field variable defined on Ω. Let (60)
$$\phi = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y},$$
and let α and β be two given positive constants. The governing equations for $\boldsymbol{u}$ on Ω are given by
$$\alpha \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) + \beta \frac{\partial \phi}{\partial x} = 0,$$
$$\alpha \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) + \beta \frac{\partial \phi}{\partial y} = 0.$$
Develop the variational formulation, and determine the allowable essential and natural boundary conditions corresponding to the above governing equations.

2. Consider the hollow cylinder of inner and outer radii a and b, respectively, which extends to infinity along the $\pm z$ directions, and subjected to a shear traction T at the inner boundary $r = a$, with the outer boundary $r = b$ fixed as shown in Fig. 1. Assume $u_r = u_\theta = 0$, $u_z = u_z(r)$, and the body forces to be zero. (40)

   (a) Using the relations
$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right],$$
$$\epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r \right), \qquad \epsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \right),$$
$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}, \qquad \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right),$$
and the constitutive relation for an isotropic material $\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$, determine the nonzero stress component/components.

   (b) Using the relations
$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r},$$
$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta = \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r},$$
$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z = \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r},$$

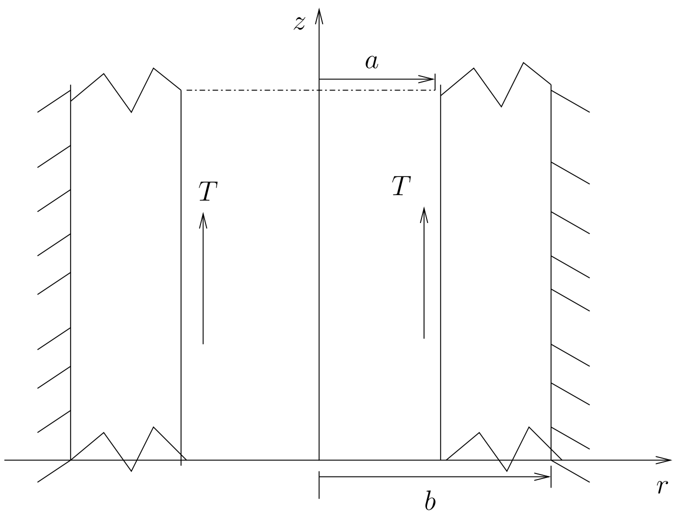

Figure 1: An infinite hollow cylinder subjected to a shear traction T at the inner boundary $r = a$, and with outer boundary $r = b$ fixed.

determine the nontrivial governing equation/equations (i.e., one or more which are not of the form $0 = 0$).

(c) Develop the variational formulation for this governing equation/equations on the domain bounded by the concentric circles $r = a$ and $r = b$, which represents the cross section of the hollow cylinder. (Hint: Combine two or more terms under one derivative sign.) In this part work only with stress components without substituting the stress-displacement relation.

(d) Substitute the stress-displacement relation that you have derived earlier into the variational formulation.

(e) Discretize u_z using the two-node element shape functions. Derive the strain-displacement matrix $\boldsymbol{B}$, and state the element stiffness matrix $\boldsymbol{K}$ in terms of $\boldsymbol{B}$. *Do not* carry out the integrations in the expression for $\boldsymbol{K}$; the integrals should be stated in terms of the natural coordinate ξ.

(f) Assuming that the components of $\boldsymbol{K}$ are K_{11}, K_{12} etc., and using a single 2-node element, determine the unknown nodal displacements in terms of these components after incorporating the appropriate boundary conditions.

(g) Derive the analytical solution for $u_z(r)$.