ME261: Assignment 2

Due: 21/9/16

1. Show that for $\boldsymbol{S} \in \text{Sym}$, the spectral decompositions of $\boldsymbol{S}^2$ and $\boldsymbol{S}^{-1}$ (when it exists) are

$$\begin{aligned}\boldsymbol{S}^2 &= \lambda_1^2 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2^2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3^2 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*,\\ \boldsymbol{S}^{-1} &= \lambda_1^{-1} \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2^{-1} \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3^{-1} \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*.\end{aligned}$$

2. The components of a symmetric tensor corresponding to a given basis are given by

$$\boldsymbol{\tau} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -3\sqrt{3} \\ 0 & -3\sqrt{3} & -5 \end{bmatrix}$$

Find the principal values and principal directions of this tensor, and find the components of the tensor with respect to the principal basis (*without* using the results of the theorem).

3. Show that

$$\begin{aligned}\boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla} \boldsymbol{u})\boldsymbol{v}] &= (\boldsymbol{\nabla} \boldsymbol{u})^T : \boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u})],\\ \boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}),\\ \boldsymbol{\nabla} \cdot (\phi \boldsymbol{v}) &= \phi(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \phi),\\ \boldsymbol{\nabla} \cdot (\boldsymbol{T}^T \boldsymbol{v}) &= \boldsymbol{T} : \boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{T}),\\ \boldsymbol{\nabla} \cdot (\phi \boldsymbol{T}) &= \phi \boldsymbol{\nabla} \cdot \boldsymbol{T} + \boldsymbol{T} \boldsymbol{\nabla} \phi,\\ \boldsymbol{\nabla}(\phi \boldsymbol{T} \boldsymbol{v}) &= \phi \boldsymbol{\nabla}(\boldsymbol{T} \boldsymbol{v}) + (\boldsymbol{T} \boldsymbol{v}) \otimes \boldsymbol{\nabla} \phi,\\ \boldsymbol{\nabla}^2(\boldsymbol{u} \cdot \boldsymbol{v}) &= \boldsymbol{u} \cdot \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}^2 \boldsymbol{u} + 2 \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v}.\end{aligned}$$

4. If $\boldsymbol{u} = \boldsymbol{x} / |\boldsymbol{x}|^3$, evaluate $\boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u}$.

5. If $\boldsymbol{x}$ is the position vector of a point, and $\boldsymbol{t}$ is the axial vector of $(\boldsymbol{T} - \boldsymbol{T}^T)$, show that

$$\int_V [\boldsymbol{x} \times (\boldsymbol{\nabla} \cdot \boldsymbol{T}) + \boldsymbol{t}] \, dV = \int_S \boldsymbol{x} \times (\boldsymbol{T} \boldsymbol{n}) \, dS.$$

6. If $\boldsymbol{Q}$ is an orthogonal tensor, show that $\dot{\boldsymbol{Q}} \boldsymbol{Q}^T$ is a skew-symmetric tensor.

7. Find the SVD of the following matrices

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$