Indian Institute of Science

ME 261: Final Exam

Date: 7/12/16.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

Solve $(1, 2, 3, 4, 5, 6)$, and $(7, 8, 9)$ as two 'bunches' so that it is easier for us to grade.

1. Let $\boldsymbol{S}$ be symmetric and positive definite (in three-dimensional space) and (10)

   (a) $\boldsymbol{S}_d = \boldsymbol{W}\boldsymbol{S}\boldsymbol{W}^T$, $\boldsymbol{W} \in \text{Skw}$.

   (b) $\boldsymbol{S}_d = \boldsymbol{A}^{-1}\boldsymbol{S}\boldsymbol{A}$, $\boldsymbol{A}$ invertible and such that $\boldsymbol{S}_d$ is symmetric.

   Determine in each of the above cases whether $\boldsymbol{S}_d$ is positive definite. *Justify* your answers.

2. Prove that $\boldsymbol{T} : \boldsymbol{S} = 0$ for all $\boldsymbol{S} \in \text{Sym}$ if and only if $\boldsymbol{T} \in \text{Skw}$. (10)

3. Let V denote the domain, S its surface and $\boldsymbol{n}$ the normal to the surface S. Further, let $\boldsymbol{b}(\boldsymbol{x})$ and $\boldsymbol{T}(\boldsymbol{x})$ be vector- and tensor-valued fields on V, respectively, and $\boldsymbol{t} = \boldsymbol{T}\boldsymbol{n}$ be a vector-valued field defined on the surface S. By integrating the equations (20)

$$(\boldsymbol{\nabla}_x \cdot \boldsymbol{T}) \otimes \boldsymbol{x} + \boldsymbol{b} \otimes \boldsymbol{x} = \boldsymbol{0},$$

   over the domain V, and with the use of the divergence theorem, find a relation between $\int_V \boldsymbol{T}\, dV$, $\int_S \boldsymbol{t} \otimes \boldsymbol{x}\, dS$ and $\int_V \boldsymbol{b} \otimes \boldsymbol{x}\, dV$.

   For the domain shown in Fig. 1, given that $t_x = t_y = 0$ on $z = 0$, $t_z = (x^2 + y^2 - a^2)$ on $z = 0$, $\boldsymbol{t} = 0$ on all the remaining surfaces, and $\boldsymbol{b} = -b_0\boldsymbol{e}_z$, find using the relation that you have derived above $\int_V T_{zz}\, dV/V_0$, where V_0 is the volume of the cylinder.

4. Solve the following first order differential equation (an implicit solution is acceptable): (10)

$$\frac{dy}{dx} = -\frac{x^3y^4 + 2x}{x^4y^3 + 4y}.$$

5. Given that $y_1 = x + 1$ and $y_2 = 1/x$ are the complementary solutions of (10)

$$x(1 + 2x)y''(x) + 2(1 + x)y'(x) - 2y(x) = 2x.$$

   find the general solution of the above equation. Do not use the formulae from the notes directly. You need not evaluate any integrals that arise, but the integrands should be stated *explicitly*.

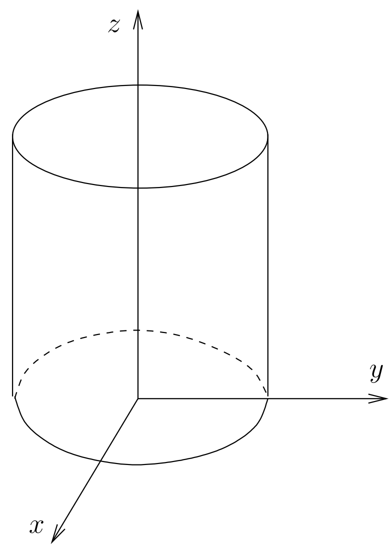

Figure 1: Domain V which is circular cylinder of radius a and height L with the z-axis along the axis of the cylinder.

6. This question leads you through an alternative way of solving a nonhomogeneous system of ordinary differential equations with constant coefficients. Let the system of equations be given by (20)

$$\boldsymbol{v}' + \boldsymbol{A}\boldsymbol{v} = \boldsymbol{f}(x), \tag{1}$$

where $\boldsymbol{A}$ is an $n \times n$ (constant) matrix, and the prime denotes differentiation with respect to x. Let $\boldsymbol{A}$ be such that it possesses n linearly independent eigenvectors $\boldsymbol{u}_i$ (not necessarily normalized to unit magnitude) corresponding to the eigenvalues λ_i, $i = 1, 2, \ldots, n$.

(a) Verify whether

$$\boldsymbol{v}(x) = \sum_{i=1}^{n} c_i e^{-\lambda_i x} \boldsymbol{u}_i,$$

where the c_i are constants, is a solution to the *homogeneous* form of Eqn. (1), i.e., $\boldsymbol{f} = \boldsymbol{0}$.

(b) The above homogeneous solution can be written in matrix form as $\boldsymbol{v}(x) = \boldsymbol{X}(x)\boldsymbol{c}$, where $\boldsymbol{X}$ is a matrix with $e^{-\lambda_i x}\boldsymbol{u}_i$ along the columns, and $\boldsymbol{c}$ is a vector of constants, i.e.,

$$\boldsymbol{X} = \left[e^{-\lambda_1 x}\boldsymbol{u}_1 \mid e^{-\lambda_2 x}\boldsymbol{u}_2 \mid \ \ldots \ \mid e^{-\lambda_n x}\boldsymbol{u}_n \right], \quad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ \cdots \\ c_n \end{bmatrix}.$$

(c) Use the method of variation of parameters whereby you write $\boldsymbol{v} = \boldsymbol{X}(x)\boldsymbol{z}(x)$, and find a general solution to Eqn. (1) in terms of $\boldsymbol{X}$, $\boldsymbol{f}$ and $\boldsymbol{c}$, where $\boldsymbol{c}$ is a vector of constants as in the above equation.

(d) Noting the fact that the above procedure works even when the eigenvalues are complex-valued, use the above solution that you have derived to solve Eqn. (1) when

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} e^{2x} \\ e^{x} \end{bmatrix}.$$

You *need not* evaluate any integrals that arise (again state the integrand explicitly).

7. Using a suitable contour evaluate the following integral (show all the steps and state all the theorems used clearly): (7)

$$I = \int_0^\infty \frac{x^{1/3}}{(x+2)(x+1)} dx.$$

8. Using a suitable contour evaluate the following integral (show all the steps clearly and state all the theorems used): (7)

$$I = \int_{-\infty}^{\infty} \frac{\cos\ x - \cos\ a}{x^2 - a^2} dx, \quad \text{a is real.}$$

9. Given the Bromwich integral (6)

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds, \quad \text{find} \quad f(t) = \mathcal{L}^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\}.$$