Indian Institute of Science

ME 261: Final Exam

Date: 3/12/18.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

Solve $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 9)$ as three 'bunches' so that it is easier for us to grade.

1. If $\boldsymbol{w}$ is the axial vector of $\boldsymbol{W} \in \text{Skw}$, find an expression for $\textbf{cof}\, \boldsymbol{W}$ in term of $\boldsymbol{w}$. (14)

   (a) Deduce an expression for $e^{\textbf{cof}\, \boldsymbol{W}}$ (the final expression should have a finite number of terms).

   (b) Find the trace and determinant of $e^{\textbf{cof}\, \boldsymbol{W}}$.

   Justify all steps in your derivation.

2. Let (8)

$$\boldsymbol{x} = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t),$$

   where $\boldsymbol{X}$ and $\boldsymbol{c}(t)$ are vectors, and $\boldsymbol{Q}(t) \in \text{Orth}^+$. Find expressions for

$$\begin{aligned}
\boldsymbol{v}(\boldsymbol{x}, t) &= \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}}, \\
\boldsymbol{F}(\boldsymbol{X}, t) &= \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right)_{t}, \\
\boldsymbol{E}(\boldsymbol{X}, t) &= \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right), \\
\bar{\boldsymbol{E}}(\boldsymbol{x}, t) &= \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right), \\
\boldsymbol{L}(\boldsymbol{x}, t) &= \boldsymbol{\nabla}_x \boldsymbol{v}, \\
\boldsymbol{D}(\boldsymbol{x}, t) &= \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right), \\
\boldsymbol{W}(\boldsymbol{x}, t) &= \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^T\right).
\end{aligned}$$

3. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors. (18)

   (a) Find a relation between $\boldsymbol{\nabla} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})]$, $\boldsymbol{\nabla} \times \boldsymbol{a}$, $\boldsymbol{\nabla} \times \boldsymbol{b}$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})$.

   (b) Integrate the above relation and apply the divergence theorem to get a relation between a volume integral and a surface integral.

   Justify all steps in your derivation.

4. Find the general solution of the following ODE: (10)

$$4t^2 \frac{d^2 x}{dt^2} - 4t \frac{dx}{dt} + (4t^2 + 3)x = 0$$

5. Consider the following inhomogeneous second order ODE: (20)

$$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+(t-\frac{1}{2})(t+\frac{1}{2})x=3t^{3/2}\sin t. \qquad (1)$$

(a) Verify that $x_1 = \sin t/\sqrt{t}$ is a solution to the corresponding homogeneous equation (RHS = 0 in Eq. 1).

(b) Using this solution, find another linearly independent solution x_2 of the homogeneous equation. Show explicitly that x_1 and x_2 are linearly independent.

(c) Using x_1 and the x_2 you have determined, find the general solution to the inhomogeneous problem given by Eq. 1.

6. Find the general solution of the following ODE: (10)

$$\frac{d^2x}{dt^2}-\frac{dx}{dt}-2x=2e^{-t}.$$

7. Using a suitable contour evaluate the following integral: (10)

$$I=\int_{-2}^{1}\frac{\sqrt{(x-1)(x+2)}}{(x+3)(x+4)}dx.$$

Show all the steps clearly and state all the theorems used. Show the rigorous steps where needed.

8. Using a suitable contour evaluate the following integral: (10)

$$I=\int_{0}^{\infty}\frac{1}{x^{\frac{1}{3}}(x-1)}dx.$$

Show all the steps clearly and state all the theorems used. Show the rigorous steps where needed.

Some relevant formulae

$$w_i=-\frac{1}{2}\epsilon_{ijk}W_{jk},$$
$$W_{ij}=-\epsilon_{ijk}w_k.$$