Indian Institute of Science

ME 261: Final Exam

Date: 6/12/22.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

Solve $(1,2)$, 3 and $(4,5,6)$ as three 'bunches' so that it is easier for us to grade.

1. We are going to deal with tensors whose underlying space is 2-dimensional. Thus, all matrices are 2×2. (25)

   (a) Consider the sequence of numbers generated by the recurrence relation

   $$A_{n+1} = 2A_n + A_{n-1}, \tag{1}$$

   with $A_0 = 0$ and $A_1 = 1$. The first few numbers in this sequence are $0, 1, 2, 5, 12, \ldots$. Our goal is to find an *explicit* formula for A_n, the n'th number in this sequence, as a function of n. With this in view, write Eqn. (1) as

   $$\begin{bmatrix} A_{n+1} \\ A_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} A_n \\ A_{n-1} \end{bmatrix}, \tag{2}$$

   where you have to determine the numbers a_1, a_2, b_1 and b_2.

   One can repeat the process in Eqn. (2) by expressing $\begin{bmatrix} A_n \\ A_{n-1} \end{bmatrix}$ in terms of $\begin{bmatrix} A_{n-1} \\ A_{n-2} \end{bmatrix}$ and so on until we get

   $$\begin{bmatrix} A_{n+1} \\ A_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^n \begin{bmatrix} A_1 \\ A_0 \end{bmatrix}, \tag{3}$$

   where $A_0 = 0$ and $A_1 = 1$.

   (b) Simplify (and this is the key step) Eqn. (3) using appropriate tensor principles.

   (c) Let $\boldsymbol{S} \in \mathrm{Sym}$. Assuming $\lambda_1 \neq \lambda_2$, and using the relations

   $$\boldsymbol{I} = \sum_{i=1}^{2} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*,$$
   $$\boldsymbol{S} = \sum_{i=1}^{2} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*,$$

   solve for $\boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^*$ and $\boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^*$ in terms of $\boldsymbol{S}$.

   (d) Combine the results of Items 1b and 1c to obtain an expression for

   $$\begin{bmatrix} A_{n+1} \\ A_n \end{bmatrix} \text{ in terms of } \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \text{ and } \begin{bmatrix} A_1 \\ A_0 \end{bmatrix}.$$

(e) Give an explicit formula for A_n in the form

$$A_n = \frac{\alpha^n - \beta^n}{\gamma}, \qquad (4)$$

where you have to determine α, β and γ.

2. Let $\boldsymbol{x}$ denote the position vector and t denote time, and let $\boldsymbol{g}(\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} - ct)$ be a vector-valued function of $(\boldsymbol{x}, t)$, where $c > 0$ is a constant. Find the value of the constants k_1 and k_2 such that (15)

$$\boldsymbol{\nabla}^2 \boldsymbol{g} = k_1 \frac{\partial^2 \boldsymbol{g}}{\partial t^2} + \frac{k_2}{|\boldsymbol{x}|}\frac{\partial \boldsymbol{g}}{\partial t}.$$

3. Consider a 1-degree of freedom oscillator - a mass (m) attached to a wall via a spring (constant k) and damper (coefficient c). The mass is released at 0 velocity from an initial distance of x_0 at $t = 0$. (20)

   (a) Write down the governing equations in terms of m, k, c, reduce it to an equation containing two constants $\omega_n = \sqrt{k/m}$ and ζ. Determine the motion of the mass $x \equiv x(t; x_0)$

   (b) If the oscillator is forced by an external force $F = F_0 \sin(\omega_f t)$, what will be the new solution? Assume the initial conditions are unchanged.

   (c) Qualitatively plot the oscillation amplitude as a function of ω_f/ω_n.

4. For the complex function

$$f(z) = \frac{\pi \sin(\pi z)}{z^3},$$

- find the order of the pole at $z = 0$. (5)
- find the residue at $z = 0$. (5)

5. Using a suitable contour evaluate the following integral : (20)

$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{(x^2-4)} dx.$$

Show all the steps clearly and state all the theorems used. Show the rigorous steps where needed.

6. Given the Bromwich integral (10)

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds, \quad \text{find} \quad f(t) = \mathcal{L}^{-1}\{\frac{s^2 - a^2}{(s^2 + a^2)^2}\}.$$

A few useful theorems.

- A function f has a pole of order k at $z = z_0$, iff

$$\lim_{z \to z_0} f(z)(z - z_0)^k$$

is finite and also nonzero.

- if f has a simple pole at $z = z_0$, then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z_0)$$

- if f has a simple pole of order k at $z = z_0$, then

$$\mathrm{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z_0)$$