Indian Institute of Science
ME 261: Midsemester Test

Date: 3/10/15.
Duration: 2.30 p.m.–4.00 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{n}$ be a unit vector. Find a relation between $[\boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{u}, \boldsymbol{n} \times \boldsymbol{v}]$ and $[\boldsymbol{u}, \boldsymbol{n}, \boldsymbol{v}]$. (20)

2. Let $\boldsymbol{S} = \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* - 4(\boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*)$, where $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*\}$ are orthonormal. Find the factors $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$ in the polar decomposition of $\boldsymbol{S}$ in terms of $\boldsymbol{e}_1^*$ and $\boldsymbol{I}$. (20)

3. Let $\boldsymbol{S} \in \mathrm{Sym}$ be invertible, and let $(\lambda_i, \boldsymbol{e}_i^*)$, $i = 1, 2, 3$, be the eigenvalues/eigenvectors of $\boldsymbol{S}$. (30)

   (a) Find an expression for $\mathbf{cof}\,(\boldsymbol{S}^{-1})$ in terms of $\boldsymbol{S}$ and its invariants.

   (b) Using this expression, find the eigenvalues/eigenvectors of $\mathbf{cof}\,(\boldsymbol{S}^{-1})$.

   (c) Using the above results determine the following: (i) If $\boldsymbol{S}$ is positive definite, does it imply that $\mathbf{cof}\,(\boldsymbol{S}^{-1})$ is positive definite? (ii) Conversely, if $\mathbf{cof}\,(\boldsymbol{S}^{-1})$ is positive definite, does it imply that $\boldsymbol{S}$ is positive definite? *Justify* your results by providing an example if the result is not true.

4. Let $\phi(\boldsymbol{x})$ be a function of position $\boldsymbol{x}$, and let (30)

$$\boldsymbol{\tau} = (\boldsymbol{\nabla}^2 \phi)\boldsymbol{I} + \alpha \boldsymbol{\nabla}(\boldsymbol{\nabla}\phi). \tag{1}$$

Determine α so that

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} = \boldsymbol{0}.$$

Substitute the value of α that you have found in Eqn. (1), and determine $\mathrm{tr}\,\boldsymbol{\tau}$ in terms of ϕ. Substitute this equation into

$$\boldsymbol{\nabla}^2(\mathrm{tr}\,\boldsymbol{\tau}) = 0,$$

to get a governing equation for ϕ. Determine if $\phi = \boldsymbol{x} \cdot \boldsymbol{x}$ satisfies this governing equation that you have determined.

Some relevant formulae

$$(\mathbf{cof}\,\boldsymbol{T})^T \boldsymbol{T} = (\det \boldsymbol{T})\boldsymbol{I}.$$