Indian Institute of Science
ME 261: Midsemester Test

Date: 1/10/16.
Duration: 2.30 p.m.–4.00 p.m.
Maximum Marks: 100

1. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the eigenvalues of $\boldsymbol{T}$ (which is not necessarily symmetric). Let (30)

$$P_1 = \lambda_1 + \lambda_2 + \lambda_3,$$
$$P_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$P_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3.$$

Show that if any invariant can be expressed in terms of the principal invariants $I_1 = \operatorname{tr} \boldsymbol{T}$, $I_2 = 1/2[(\operatorname{tr} \boldsymbol{T})^2 - \operatorname{tr}(\boldsymbol{T}^2)]$ and $I_3 = \det \boldsymbol{T}$, it can also be expressed in terms of P_1, P_2 and P_3. *Derive* any formula for $\det \boldsymbol{T}$ that you require in terms of $\boldsymbol{T}$ by using the Cayley–Hamilton theorem

$$\boldsymbol{T}^3 - I_1 \boldsymbol{T}^2 + I_2 \boldsymbol{T} - I_3 \boldsymbol{I} = \boldsymbol{0}.$$

2. Let (35)

$$\boldsymbol{F} = \lambda \boldsymbol{e} \otimes \boldsymbol{e} + \cos\alpha(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}) + \sin\alpha(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}),$$

where $\lambda > 0$, and $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{r}\}$ forms an orthonormal basis. Find the factors $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ in the polar decomposition of $\boldsymbol{F}$ (in terms of λ, $\boldsymbol{e}$, $\boldsymbol{q}$ and $\boldsymbol{r}$). You may directly use formulae related to dyadic products such as the formulae for $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d})$ etc. Using the polar decomposition, and assuming $\boldsymbol{R}$ to be proper orthogonal, find $\det \boldsymbol{F}$ (Hint: Find the components of $\boldsymbol{U}$ with respect to a convenient basis that will allow finding its determinant easily).

3. Let $\boldsymbol{Q} \in \text{Orth}^+$ be a *constant* rotation (not dependent on $\boldsymbol{x}$ or $\boldsymbol{x}^*$). Let (35)

$$\boldsymbol{x}^* = \boldsymbol{Q}\boldsymbol{x},$$
$$\boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T.$$

Find an expression for $\boldsymbol{\nabla}_{\boldsymbol{x}^*} \cdot \boldsymbol{T}^*$ (which denotes the divergence of $\boldsymbol{T}^*$ with respect to $\boldsymbol{x}^*$) in terms of $\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{T}$ (which denotes the divergence of $\boldsymbol{T}$ with respect to $\boldsymbol{x}$) and $\boldsymbol{Q}$.