Indian Institute of Science
ME 261: Midsemester Test

Date: 2/10/22.
Duration: 9.30 a.m.–11.30 a.m.
Maximum Marks: 100

1. A rotation can be represented as (30)

$$\boldsymbol{R} = \boldsymbol{e} \otimes \boldsymbol{e} + \cos\alpha(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}) + \sin\alpha(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}),$$

where $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{r}\}$ forms an orthonormal basis. Determine the eigenvalues of $\boldsymbol{R}$ in terms of α. Find the axis of $\boldsymbol{R}$ (guesswork allowed). Determine the values of α for which $\boldsymbol{R} = \boldsymbol{I}$ and for which $\boldsymbol{R} \in \mathrm{Sym} - \{\boldsymbol{I}\}$. When $\boldsymbol{R} \in \mathrm{Sym} - \{\boldsymbol{I}\}$ find the eigenvectors of $\boldsymbol{R}$ in terms of $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{r}\}$ (if nonunique, one choice is fine).

2. Let $\boldsymbol{H} := \mathbf{cof}\,(\boldsymbol{W} - \alpha\boldsymbol{W}^3)$, where $\alpha > 0$, and $\boldsymbol{W}$ is a skew-symmetric tensor whose axial vector $\boldsymbol{w}$ is of unit magnitude. Determine $\boldsymbol{H}$ in terms of the axial vector $\boldsymbol{w}$ and α. Next find $(\boldsymbol{I} + \boldsymbol{H})^{-1}$ in terms of $\boldsymbol{w}$ and α. Justify all steps. (40)

3. Let $\boldsymbol{W}(\boldsymbol{x}, t) \in \mathrm{Skw}$, and let $\boldsymbol{w}(\boldsymbol{x}, t)$ be its axial vector. (30)
   (a) Find an expression for $\boldsymbol{\nabla} \times \boldsymbol{W}$ in terms of the gradient, divergence etc. of $\boldsymbol{w}$.
   (b) Let $\boldsymbol{W} = \boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T$ be a skew-symmetric tensor. Find the axial vector of $\boldsymbol{W}$ in terms of the curl of $\boldsymbol{u}$. Use the expression that you have derived in part (a) to find $\boldsymbol{\nabla} \times [\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T]$ in terms of curl of $\boldsymbol{u}$. Simplify this expression until no further simplification is possible.

Some relevant formulae

$$\begin{aligned}
w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk}, \\
W_{ij} &= -\epsilon_{ijk}w_k, \\
(\mathbf{cof}\,\boldsymbol{T})_{ij} &= \frac{1}{2}\epsilon_{imn}\epsilon_{jpq}T_{mp}T_{nq}, \\
I_2(\boldsymbol{T}) = \mathrm{tr}\,(\mathbf{cof}\,\boldsymbol{T}) &= \frac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{T})^2 - \mathrm{tr}\,\boldsymbol{T}^2\right], \\
\det \boldsymbol{T} &= \epsilon_{ijk}T_{i1}T_{j2}T_{k3} = \epsilon_{ijk}T_{1i}T_{2j}T_{3k}, \\
(\mathbf{cof}\,\boldsymbol{T})^T\boldsymbol{T} &= (\det \boldsymbol{T})\boldsymbol{I}, \\
(\boldsymbol{\nabla} \times \boldsymbol{T})_{ij} &= \epsilon_{irs}\frac{\partial T_{js}}{\partial x_r}, \\
x^3 - ax^2 + ax - 1 &= (x-1)(x^2 + (1-a)x + 1)
\end{aligned}$$