ME304: Applied Maths for Mechanics
Part I: Complex functions
Problem Set February 21, 2023

Many of these questions are open-ended. Some might lead to fruitful research problems. As always, they draw on material discussed during our meetings but take things a few notches higher. If you find yourself tied up in knots, please use one of the many references cited during our meetings. They may provide useful clues, even though these problems were made up by us.

Problem 1: Visualizing complex maps

The aim of this exercise is to visualize standard complex functions. Use four methods of visualization:

- Using the conjugate vector field (CVF) representation.
- Using the hue-brightness scheme. You might find `https://pypi.org/project/cplot/` useful.
- By the old-fashioned route of mapping individual coordinate curves x = const., y =const. to their counterparts in the u, v plane
- By mapping simple loops (e.g., circle of various radii/centres) in the z-plane to the Z-plane

Do this for the functions $f(z) = \sin(z), \cos(z), \tan(z), \exp(z), \ln(z)$ as well as for the polynomials $(z - z_0)^n$ for $n = \ldots, -2, -1, 1, 2 \ldots$.

Problem 2: Some algebra and series expansions

It is useful to be able to do algebra once in a while (it'll also help in Part II). Obtain Laurent series expansions for the following functions about the specified points and discuss the nature of singularities at these points. Also take special care about the validity of the series:

(a) $f(z) = [z(z-i)(z^2-1)]^{1/2}$ about $z = -i$

(b) $f(z) = z^{1/2}\,[1 + \sin z]^{-1}$ about $z = -\pi/2$

(c) $f(z) = \exp\left[t(z + z^{-1})\right]$ about origin

Problem 3: Inside or out?

Find an unsuspecting friend (preferably incompetent with winding numbers) and ask them to draw 'the most horrible multiple self-intersecting closed loop' they can imagine on a piece of paper. Now ask them to pick a point P randomly on the paper and seemingly caught up in said horrible loop (called δ) but not *on* it. Based on our discussions of winding numbers, devise a method to determine if P is inside or outside δ.

Problem 4: Biharmonic equation

We will now attempt to obtain singular solutions of the biharmonic equation

$$\nabla^4 \psi = 0 \qquad (1)$$

for various singular situations that we've discussed.

(a) Starting from the full Navier-Stokes equations for a 2D problem, obtain Eq. 1 for the streamfunction (Stokes' flow problem). You want to systematically non-dimensionalize the equation and set $Re \to 0$ in a self-consistent manner. Alternatively, obtain the same equation using the equations of linear elasticity, coupled with the conditions of elastic compatibility.

(b) Convert to $z, \bar{z}$ from x, y and obtain the general solution $\psi = \mathrm{Re}(\bar{z}f(z) + g(z))$. Using this relation, derive expressions for the pressure (trace of stress tensor), velocity (displacement) and net force for the viscous (elastic) problem.

(c) Based on these relations and using logical reasoning for jumps in various quantities, deduce expressions for $f(z), g(z)$ for the following: Stokeslet (viscous), edge dislocation (elastic), point force (both).

(d) How would you handle a boundary condition in each case? A sample calculation, as discussed in class, is provided in the paper by DG Crowdy & Y Or *Phys. Rev. E*, (2010).

Problem 5: Biharmonic elastic wedge

For the wedge problem we discussed in class, can you reconcile the three routes for obtaining the stress function Ψ from the biharmonic equation? These are:

1. Expanding the general solution $\Psi = \mathrm{Re}(\bar{z}f(z) + g(z))$ in polar coordinates with $z = r\exp(-i\theta)$
2. Separation of variables in r, θ coordinates
3. The scaling solution, with $\Psi = (\epsilon/r)^{\lambda}\Phi(\theta, \alpha)$?

As you'd expect, the solutions from part (a) to (c) represent increasing seriousness of breakdown. Can you explain what they are?