ME304: Applied Maths for Mechanics
Part II: Asymptotic expansions
Problem Set 2 March 18, 2023

The following problems are based on the method of matched asymptotic expansions.

Problem 1: Ordering breakdown

Determine a two-term asymptotic solution for the equation

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \epsilon \left[\left(\frac{dy}{dx} \right)^2 + y \right] = \epsilon \exp(-2x) \tag{1}$$

on the unbounded domain $x \geq 0$ and boundary conditions $y(0) = 2, \frac{dy}{dx}(0) = -1$ and $\frac{d^2y}{dx^2}(0) = 1$. Where does ordering breakdown for the two-term expansion? What scaling do you expect for the 'layer' solution?

Problem 2: Two term expansions

Determine a two-term expansion for the following boundary value problem:

$$\epsilon \frac{d^2y}{dx^2} + \sqrt{x}\frac{dy}{dx} + y^2 = 0 \qquad x \in [0,1] \tag{2}$$

subject to the boundary conditions $y(0) = 2$ and $y(1) = 1/3$ for $\epsilon \to 0$. Plot the numerical solution of this problem alongside the expansion you've determined above. Can you compute the maximum error inside the domain for $\epsilon = 0.1$ and $\epsilon = 0.001$?

Problem 3: Multiple layers

Consider the boundary value problem:

$$\epsilon \frac{d^4y}{dx^4} - \left(\frac{1}{1 - \frac{3x}{4}} \right) \frac{dy}{dx} - 8 = 0 \qquad x \in [0,1] \tag{3}$$

subject to $y(0) = \frac{dy}{dx}(0) = y(1) = \frac{dy}{dx}(1) = 0$. Determine a composite expansion for $y(x, \epsilon)$ as $\epsilon \to 0$. As in the previous question, compare your solution against a numerical evaluation for $\epsilon = 0.1, 0.01$.

Problem 4: Turning layers

For the equation

$$\epsilon^3 \frac{d^2y}{dx^2} + (x - 2x^2)y = 0 \qquad x \in [0,1] \tag{4}$$

with boundary conditions $y(0) = \alpha, y(1) = \beta$, determine the location of transition layers and obtain the leading order term after matching.

Problem 5: Laminar flow through a porous medium

You are asked to evaluate the flow through a porous parallel plate channel of width $2h$. Assume that the Reynolds number is very high so that $\epsilon = 1/Re \to 0$.

(a) Set up the problem assuming incompressible flow for a 2D flow field $\vec{u} = (u(x,y), v(y))$ and boundary conditions $u(x, y = \pm h) = 0$ and $v(y = \pm h) = V_\pm$. Assume that $V_+ > V_-$.

(b) Using the continuity equation, determine $u(x,y)$ and $v(y)$ in terms of an unknown function $u_0(x)$

(c) From the Navier-Stokes equations for this configuration, obtain an expression for the kinematic pressure $p(x,y) = \frac{1}{\rho}P(x,y)$.

(d) Non-dimensionalize all the equations, use V_+, h as velocity and length scales.

(e) Derive a third order ODE for $v(y)$ subject to the boundary conditions in part (a).

(f) Solve for $v(y)$ by evaluating possible combinations of values for $V_\pm$. Where does a layer occur in this problem?

If you are stuck, you might want to look at the reference: I Proudman, *J Fluid Mech*, 1960, where this problem was first presented.

Problem 6: Heating of a current carrying insulated wire

Renovation of the department workshop has led to you questioning the need for large diameter (expensive) wires for high current applications. Consider a cylindrical wire of length L and radius R which has an outer insulation layer of thickness $R_0 - R$. The potential difference between the ends of the wire is V. Given that it has to carry large currents, the electrical σ and thermal K conductivities of the conducting part depend on temperature as:

$$\sigma = \sigma_0(1 - B(T - T_0)) \qquad K = K_0(1 - A(T - T_0)) \tag{5}$$

the insulator layer has constant conductivity κ. The outer surface of the insulator cladding is maintained at temperature T_0, which also happens to be the reference temperature in Eq. 5.

(a) Set up the problem using the heat conduction equation in both the core wire and the insulator. Remember that a volumetric heat source is operational inside the conducting part of the wire due to the current. Obtain an expression for this heat source.

(b) What are the boundary conditions at $r = R, r = R_0$?

(c) Non-dimensionalize the entire system using $\xi = r/R$ and $\theta = (T - T_0)/T_0$. The heating term should be now $1/J$. What is J?

(d) Solve the problem for very large J (what is the physical significance?) using dominant balance

(e) Now solve the more difficult complementary problem for very small $J \equiv \epsilon$. Obtain two-term asymptotic expansion for $T(r)$ in the core of the wire. Physically compare your results for $J \gg 1$ and $J \ll 1$.

If you solved this problem in the year 1970, you could have published your result. For reference, you can find the solution to this problem in: Cohen and Shair, *Int J Heat Mass Transfer*, 1970.