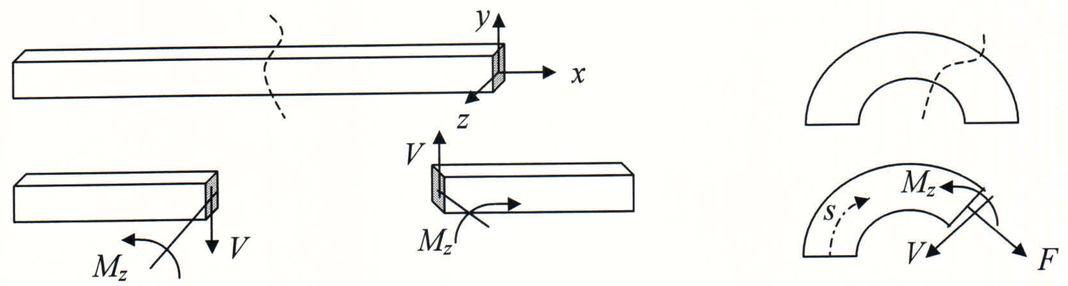
Beams – Review of straight beams and introducing curved beams

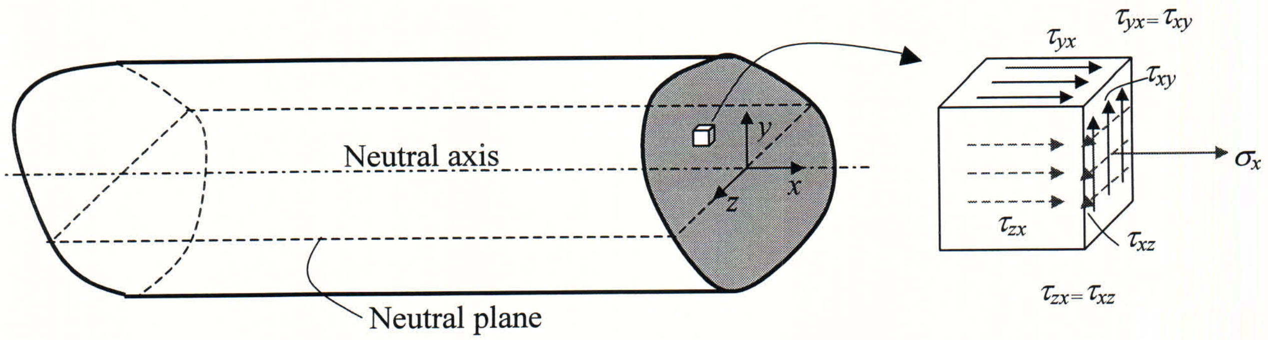
Beams are one-dimensional models of slender 3-D elastic continua. They can be subject to axial forces like bars and their behavior is same as bars in this case. Now, we will study the effects of loads perpendicular to the axis, called transverse loads. Transverse loads can act either in the y direction or z direction for a beam shown in the figure below. The effect of transverse loads is to bend the beam. Loads in the y direction will bend it in the xy-plane while the z-loads bend it in the xz-plane. In what follows, we will only consider transverse loads in the y direction because the effects are the same for z-loads as well. Additionally, a beam may also be subject to torque (i.e., moments about the axial direction) loads. Torque loads cause the beam to twist about its axis.

When you cut a portion of a beam and make it free-body, on the cross-section at the point where it is cut, a vertical shear force V, and a bending moment M_z will exist as reactions to keep the free-body in static equilibrium with external forces and support reactions. Our sign convection for positive values is shown in the figure. For a curved beam, the axis is curved. In it, transverse loads also create an axial reaction force F, as shown. Drawing shear force and bending moment (and axial force) diagrams for curved beams is same as it is for straight beams. The difference is only that we should move along the curved axis with a path variable s instead of x. The transverse loads are always normal (i.e., perpendicular) to the axis.

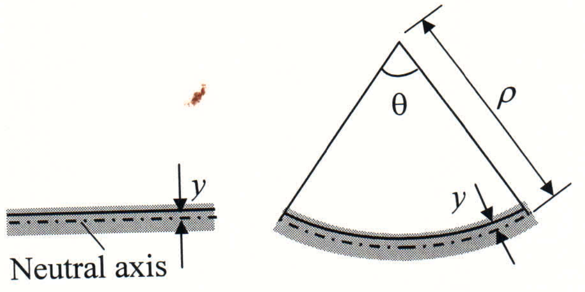


Straight beam   Curved beam

As can be imagined, when the beam bends, say concave down, the lines parallel to the axis but above it contract and the ones below it elonagte. So, there must exist a line that neither elongate nor contracts. This is called the *neutral axis* and taken as the axis of the beam. The zx-plane containing the neutral axis is called the *neutral plane*. See the figure below. Since beam is a one-dimensional model, we only consider stresses that involve the axial direction, i.e., σ_x, τ_{xy}, and τ_{xz}. These are shown for an infinetesimally small cube taken inside a beam.



Straight beam in pure bending



Before bedning — After

Axial strain = $\varepsilon_x = \dfrac{(\rho-y)\theta - \rho\theta}{\rho\theta} = \dfrac{-y}{\rho}$

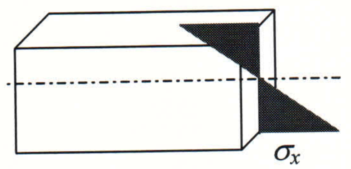
$\varepsilon_m = |\varepsilon_{max}| = \dfrac{c}{\rho}$

c = max. absolute distance from the neutral axis of the farthest point on c/s.

$\rho = \dfrac{c}{\varepsilon_m} \Rightarrow \varepsilon_x = -\dfrac{y}{c}\varepsilon_m$

$\Rightarrow \sigma_x$ = axial stress = $-\dfrac{y}{c}E\varepsilon_m = -\dfrac{y}{c}\sigma_m$

σ_x is linear in y.



where is the neutral axis?

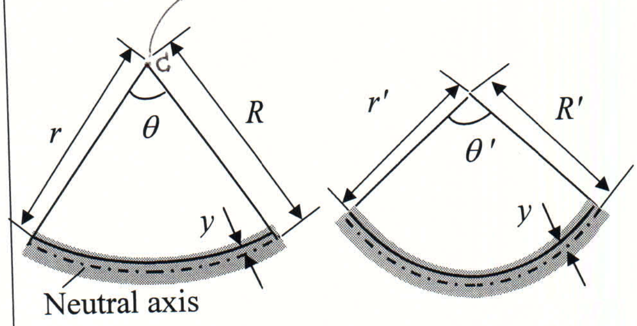
comes from force-balance in the x direction at the c/s.

$\Sigma F_x = 0 \Rightarrow \int \sigma_x \, dA = 0$

$\Rightarrow -\dfrac{\sigma_m}{c}\underbrace{\int y \, dA}_{} = 0$

This is true only if the neutral axis passes through the centroid of the c/s.

Curved beam in pure bending



Before bedning — After

$\theta' = \theta + \Delta\theta$

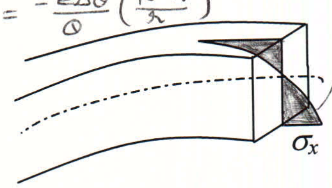
$R\theta = R'\theta'$ (because neutral axis doesn't contract or elongate)

$\varepsilon_x = \dfrac{r'\theta' - r\theta}{r\theta} = \dfrac{(R'-y)\theta' - (R-y)\theta}{r\theta}$

$= \dfrac{-y\Delta\theta}{\theta(R-y)}$ {using $R\theta = R'\theta'$}

$\sigma_x = -\dfrac{E y \Delta\theta}{\theta(R-y)} = -\dfrac{E\Delta\theta}{\theta}\left(\dfrac{R-r}{r}\right)$

σ_x is not linear in y.



arc of a hyperbo

where is the neutral axis?

$\Sigma F_x = 0 \Rightarrow \int \sigma_x \, dA = 0$

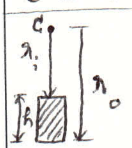
$\Rightarrow -\dfrac{E\Delta\theta}{\theta}\int \dfrac{R-r}{r}\, dA = 0$

$\Rightarrow \int \dfrac{R}{r}\, dA - \int dA = 0 \Rightarrow R = \dfrac{A}{\int \frac{dA}{r}}$

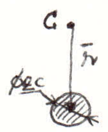
Recall that $\bar{r} = \dfrac{1}{A}\int r \, dA$

$\bar{r}$ = distance from the center of curvature to the centroid of the c/s.

$e = \bar{r} - R = \dfrac{1}{A}\int r\, dA - \dfrac{A}{\int \frac{dA}{r}}$



$R = \dfrac{h}{\ln \frac{r_o}{r_i}}$



$R = \frac{1}{2}\left(\bar{r} + \sqrt{\bar{r}^2 - c^2}\right)$

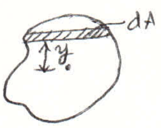
Magnitude of σ_m ?

Comes from z-moment balance.

$$\Sigma M_z = 0 \Rightarrow \int y\sigma_x \, dA + M = 0$$

$$\Rightarrow \frac{\sigma_m}{c} \underbrace{\int y^2 dA}_{} = M$$

defined as second moment of inertia of c/s area.



$$\therefore \sigma_m = \frac{Mc}{I} \Rightarrow \sigma_x = -\frac{My}{I}$$

Infact M is M_z and I is I_z.
Recall that we are only considering transverse loads in the y-direction.
If we consider z-transverse loads, there will be M_y too.
In general for a beam with y and z transverse loads,

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

This "+" is correct in our sign convention.

For a beam with 3-D loads, both M_z and M_y become relevant. If there are axial loads, $\frac{F}{A}$ also needs to be added to the mix in computing σ_x.

Isn't it wonderful that Hooke's law and static equilibrium gives so much information about beams?

Magnitude of σ_x ?

(Note we still don't know $\frac{\Delta\theta}{\theta}$).
Again, it comes from z-moment balance.

$$\Sigma M_z = 0 \Rightarrow \int y\sigma_x \, dA + M_z = 0$$

$$\Rightarrow \int \frac{E\Delta\theta}{\theta} \frac{R-r}{r} y \, dA = M_z$$

Noting that $y = R - r$,

$$\Rightarrow \frac{E\Delta\theta}{\theta}\left\{ \int \frac{(R-r)^2}{r} dA \right\} = M_z$$

$$\Rightarrow \frac{E\Delta\theta}{\theta}\left\{ \int \frac{R^2}{r} dA + \int r \, dA - \int 2R \, dA \right\} = M_z$$

$$\Rightarrow \frac{E\Delta\theta}{\theta}\left\{ RA + \bar{r}A - 2RA \right\} = M_z$$

$\because R = \frac{A}{\int \frac{dA}{r}}$ $\quad$ $\because \bar{r} = \frac{\int r \, dA}{A}$

$$\Rightarrow \frac{E\Delta\theta}{\theta} = \frac{M_z}{A(\bar{r}-R)} = \frac{M_z}{Ae}$$

Since $\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{y}{(R-y)}$, the above equation gives

$$\sigma_x = \frac{-M_z y}{Ae(R-y)} = \frac{M(r-R)}{Aer}$$

Deflection of the neutral axis

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{EC} = \frac{MC}{I}\cdot\frac{1}{EC} = \frac{M}{EI}$$

$$\underbrace{\frac{M}{EI} = \frac{1}{\rho}}_{\text{Euler-Bernoulli theorem}} = \text{curvature}$$

$$\frac{1}{\rho} = \text{curvature} = \frac{w''}{(1+w'^2)^{3/2}}$$

$w =$ transverse deflection of the neutral axis as a function of x ; $\boxed{\text{+ve downward}}$

$$w' = \frac{dw}{dx} \qquad w'' = \frac{d^2w}{dx^2}$$

For small deflections, w'^2 can be neglected.

$$\therefore \frac{1}{\rho} \simeq w''$$

$$\Rightarrow \quad w'' = \frac{M}{EI} \quad \& \quad EIw'' = M$$

Since $\frac{dM}{dx} = V$ and $\frac{dV}{dx} = f$ = distributed load,

we can write

$$M = EI\,w''$$

$$V = EI\,w''' = EI\frac{d^3w}{dx^3}$$

$$q = EI\,w^{IV} = EI\frac{d^4w}{dx^4}$$

Deflection of the neutral axis

$$R\theta = R'\theta'$$ (see what we had before)

$$\Rightarrow \frac{1}{R'} = \frac{1}{R}\frac{\theta'}{\theta}$$

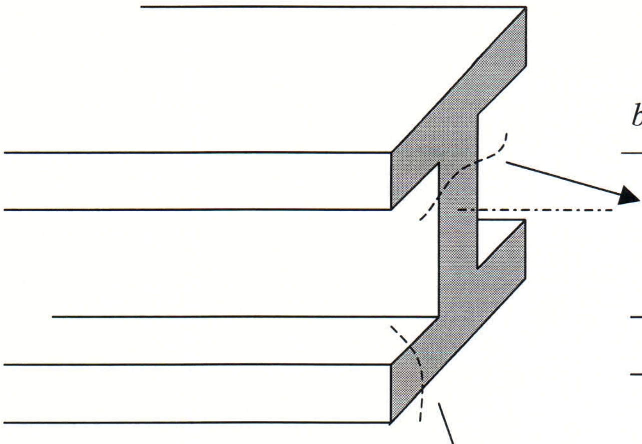
$$= \frac{1}{R}\left(\frac{\theta+\Delta\theta}{\theta}\right) = \frac{1}{R}\left(1+\frac{\Delta\theta}{\theta}\right)$$

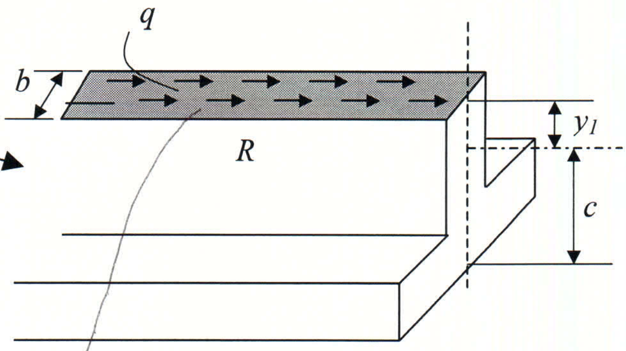
$$\frac{\Delta\theta}{\theta} = \frac{M_z}{EAe}$$

$$\Rightarrow \frac{1}{R'} - \frac{1}{R} = \frac{M}{EAeR}$$

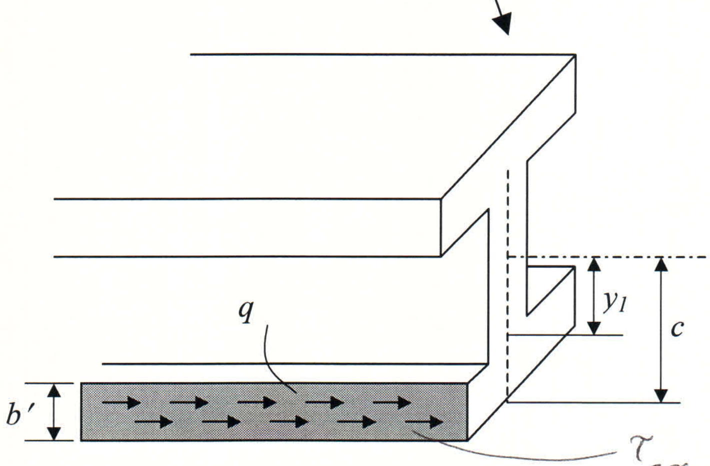
This can be used to compute the transverse deflection.

Shear stress due to bending of beams





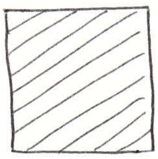
$$\tau_{xy} = \frac{VQ}{Ib} = \tau_{yx}$$



Shear flow = $q = \dfrac{VQ}{I}$

$$Q = \int_{y_1}^{c} y \, dA$$

$$\tau_{zx} = \tau_{xz} = \frac{VQ}{Ib'}$$

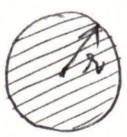


h = height

b = width

$$Q = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right) \Rightarrow \tau_{max} = \frac{3V}{2A}$$

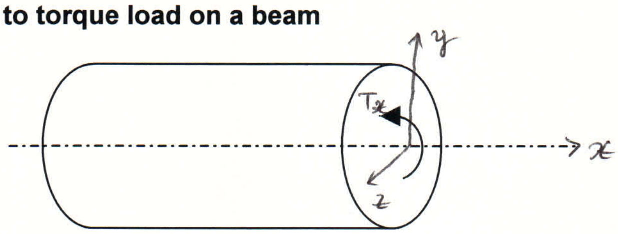


$$Q = \frac{2}{3}\left(r^2 - y_1^2\right)$$

$$\tau = \frac{4}{3}\frac{V}{\pi r^2}\left(1 - \frac{y_1^2}{r^2}\right)$$

$$\tau_{max} = \frac{4V}{3A}$$

Shear stress due to torque load on a beam



$$\tau_{xy} = \frac{T r'}{J}$$

r' = distance from center to any point on c/s

$J = \int r^2 \, dA$ = polar moment of inertia of c/s

$$= \frac{\pi d^4}{32} \text{ for a circle c/s}$$

$$\Rightarrow \tau_{max} = \frac{T r}{J}$$

r = radius of circle.

For rectangular c/s,

$$\tau_{xy} = \frac{T}{Q} \; ; \; Q = \frac{8a^2 b^2}{3a + 1.8b}$$

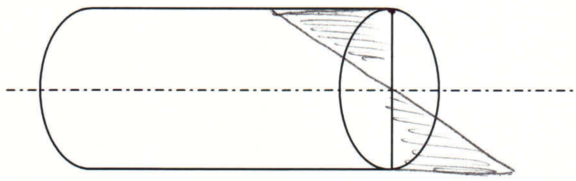
$$\boxed{G = \text{rigidity modulus} = \frac{E}{2(1+\nu)}}$$

<u>Twists</u>

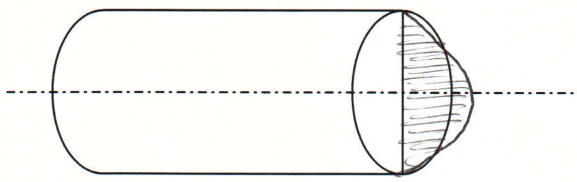
$$\theta = \frac{T l}{J G} \text{ for circular c/s}$$

$$= \frac{T l}{K G} \text{ for rect. c/s, } K = ab^3\left(\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right)$$

For a beam with circular cross-section with transverse loads and torque….

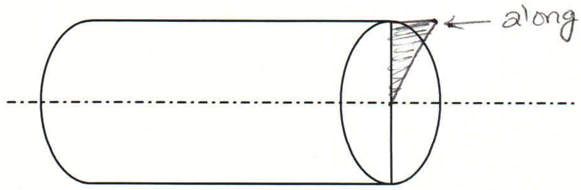


Normal stress due to bending , σ_x



Shear stress due to bending , τ_{xy}

← along all radii in all directions



Shear stress due to torsion , τ_{xy}

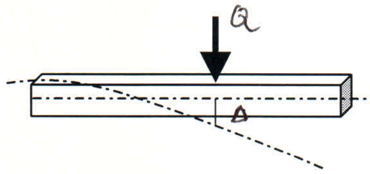
Strain energies in bars and beams

Type of loading	*Stress*	*Strain*	*Strain energy in terms of loads and material properties*	*Strain energy in terms of deflections and material properties*
Axial				
Bending moment				
Vertical shear				
Torque				

Castiglianos theorem

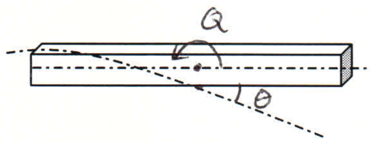
Castiglianos theorem is one of the powerful applications of energy methods and is very helpful in determining the deflections at specific points without solving for deflections everywhere. It is also very useful in statically indeterminate structures to determine unknown reaction forces because it facilitates determination of deflections at specific points without much work. The theorem states that the deflection at a point is equal to the partial derivative of the strain energy taken with respect to a dummy (or real) load applied at the point of interest in the direction of interest. Mathematically,

$$\Delta = \frac{\partial(SE)}{\partial Q}$$



Since we always imply a moment load when we say a load, the theorem can also be used to compute deflections. Then, the load Q is interpreted as a moment. The deflection determined using the theorem is the slope at the point where the moment is applied.

$$\theta = \frac{\partial(SE)}{\partial Q}$$



In order to apply this theorem, we just need to compute the strain energy symbolically in terms of the load Q. If a real load does not exist at that point, we add a dummy load. After we get the expression for the deflection (or slope), we simply substitute Q = 0. It is just as easy!