Material covered in BE 205
Jan. 8 – Feb. 15, 2013

* Equations of static equilibrium

2D	3D	
$\sum F_x = 0$	$\sum F_x = 0$	$\sum M_x = 0$
$\sum F_y = 0$	$\sum F_y = 0$	$\sum M_y = 0$
$\sum M_z = 0$	$\sum F_z = 0$	$\sum M_z = 0$

* Moment = turning effect of force

$$\vec{M} = \vec{r} \times \vec{F}$$

$$M_x \hat{i} + M_y \hat{j} + M_z \hat{k} = (r_x \hat{i} + r_y \hat{j} + r_z \hat{k}) \times (F_x \hat{i} + F_y \hat{j} + F_z \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

* Free-body diagram: a diagram showing a body that is made free from all contacts but includes all forces of interaction.
  - When two bodies are in contact, the contact force is along the common normal. The friction force is perpendicular to the normal force.

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		Centroid	Centre of mass	Centre of gravity
For N particles		$\dfrac{\sum_{i=1}^{N} \vec{r}_i}{N}$	$\dfrac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}$	$\dfrac{\sum_{i=1}^{N} w_i \vec{r}_i}{\sum_{i=1}^{N} w_i}$
Discrete set of bodies	2D	$\dfrac{\sum A_i \vec{r}_i}{\sum A_i}$	$\dfrac{\sum \rho_i t_i A_i \vec{r}_i}{\sum \rho_i t_i A_i}$	$\dfrac{\sum \rho_i g t_i A_i \vec{r}_i}{\sum \rho_i t_i g A_i}$
	3D	$\dfrac{\sum v_i \vec{r}_i}{\sum v_i}$	$\dfrac{\sum \rho_i v_i \vec{r}_i}{\sum \rho_i v_i}$	$\dfrac{\sum \rho_i g v_i \vec{r}_i}{\sum \rho_i g v_i}$

Summation is replaced by integration for continuous bodies.

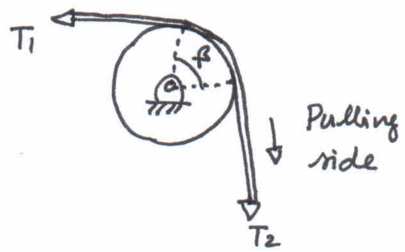
* **Friction:** $F_{friction} = \mu N$

μ ← Coefficient of friction

N ← Normal force at the contact

μ < μ_s for static condition, μ_k for moving condition

Usually, $\mu_k < \mu_s$.



$$T_2 = T_1 e^{\mu\beta}$$

μ = coefficient of friction between the pulley and the belt / rope

β = angle of overlap

$T_2 \geq T_1$

* Degrees of freedom (dof) = the minimum number of independent actuations that determine the complete confi

Grübler's formula:

$$\#\,dof = 6(n-1) - 5f_1 - 4f_2 - 3f_3 - 2f_4 - f_5 \quad \text{in 3D}$$

$$= 3(n-1) - 2f_1 - f_2 \quad \text{in 2D}$$

and so on

n = number of bodies including the fixed body

f_i = number of joints that allow ~~d~~ i relative freedoms between the bodies that the joints connect

* Loop-closure equations.

Add the vectors in the order they appear by following the directions of the vectors.

The vectors can be position vectors, velocity vectors, or acceleration vectors.

* Moving reference frames

$$\underbrace{\vec{r}_P}_{\substack{\text{Position}\\ \text{vector in the}\\ \text{absolute reference}\\ \text{frame}}} = \underbrace{\vec{r}_A}_{\substack{\text{Position}\\ \text{vector of}\\ \text{the origin}\\ \text{of the}\\ \text{moving frame relative to the absolute}\\ \text{reference frame}}} + \underbrace{\vec{r}_{P/A}}_{\substack{\text{position vector relative to}\\ \text{the origin of the moving}\\ \text{frame A}}}$$

Similarly,

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} \quad \text{and} \quad \vec{a}_P = \vec{a}_A + \vec{a}_{P/A}$$

for velocity $\qquad\qquad$ for acceleration

* Finite rotations

$$\underbrace{\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}}_{\substack{\text{Coordinates}\\ \text{after rotation}}} = \underbrace{\begin{bmatrix} \quad \bar{\bar{R}} \quad \end{bmatrix}}_{\substack{\text{Rotation}\\ \text{matrix}}} \underbrace{\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}}_{\text{Coordinates before rotation}}$$

$$\bar{\bar{R}} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Rotation about the z-axis by θ_z.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}$$ About the x-axis by θ_x.

$$= \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$ About the y-axis by θ_y.

Space-fixed multiple rotations: $\bar{\bar{R}} = \bar{\bar{R}}_1 \bar{\bar{R}}_2 \bar{\bar{R}}_3 \cdots \bar{\bar{R}}_n$

Body-fixed multiple rotations: $\bar{\bar{R}} = \bar{\bar{R}}_n \bar{\bar{R}}_{n-1} \cdots \bar{\bar{R}}_2 \bar{\bar{R}}_1$

* Finite rotations do not commute. That is $\bar{\bar{R}}_1 \bar{\bar{R}}_2 \neq \bar{\bar{R}}_2 \bar{\bar{R}}_1$. So, the order of rotations does matter.

* Rotation about an arbitrary axis:

$$\bar{\bar{R}}(\hat{a}, \phi) = \begin{bmatrix} a_x^2 v\phi + c\phi & a_x a_y v\phi - a_z s\phi & a_x a_z v\phi + a_y s\phi \\ a_x a_y v\phi + a_z s\phi & a_y^2 v\phi + c\phi & a_y a_z v\phi - a_x s\phi \\ a_x a_z v\phi - a_y s\phi & a_y a_z v\phi + a_x s\phi & a_x^2 v\phi + c\phi \end{bmatrix}$$

where

$$\hat{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \text{axis vector (preferably unit vector so that } a_x^2 + a_y^2 + a_z^2 = 1.$$

ϕ = angle of rotation about the axis

$c\phi = \cos\phi$; $s\phi = \sin\phi$; $v\phi = 1 - \cos\phi$

* Euler's theorem of rigid-body rotation:

"The most general rotation of a rigid-body can be described by a single rotation about an axis". The axis is called the principal line.

Chasles's theorem:

"The most general displacement of a rigid-body is equivalent to a translation of a point in the body plus a rotation about an ~~an~~ axis through that point." The axis is called the screw axis.

$$\underbrace{2\cos\phi}_{\text{Twist = rotation around the screw axis}} + 1 = \text{trace}(\bar{\bar{R}}) = \text{sum of the diagonal entries in } \bar{\bar{R}}$$

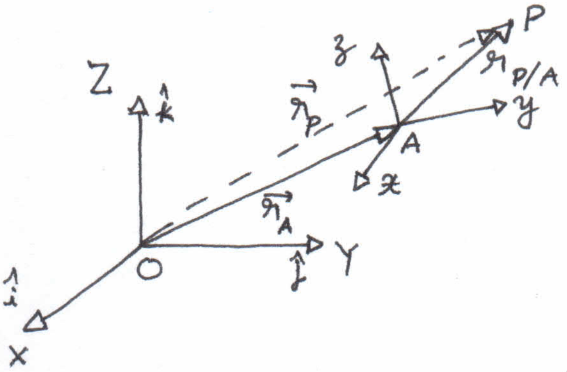
$$\bar{\bar{\omega}} = \frac{1}{2\sin\phi}\left(\bar{\bar{R}} - \bar{\bar{R}}^T\right) = \text{skew-symmetric matrix of the form } \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

From this the axis can be obtained

$$a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Note: $\bar{\bar{R}} = e^{\phi\bar{\bar{\omega}}}$

* Relative motion in rotating bodies



$$\vec{r}_P = \vec{r}_A + \vec{r}_{P/A}$$

The moving frame x-y-z is translating and rotating relative to the fixed frame X-Y-Z.

$\vec{r}_A$ = position vector of the origin A of x-y-z

$\vec{V}_A$ = velocity vector of A

$\vec{a}_A$ = acceleration vector of A

Let $\vec{\omega} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

= angular velocity vector of x-y-z frame relative to X-Y-Z frame

$$\vec{V}_P = \vec{V}_A + \underbrace{\vec{V}_{P/A}}_{\text{Relative velocity of P in } x\text{-}y\text{-}z \text{ frame}} + \vec{\omega} \times \vec{r}_{P/A}$$

$$\vec{a}_P = \vec{a}_A + \vec{\alpha} \times \vec{r}_{P/A} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/A})}_{\text{Centripetal acceleration}} + \vec{a}_{P/A} + \underbrace{2\vec{\omega} \times \vec{V}_{P/A}}_{\text{Coriolis acceleration}}$$

$\vec{\alpha}$: angular acceleration vector of x-y-z frame relative to X-Y-Z frame

$\vec{a}_{P/A}$: Relative acceleration of P in x-y-z frame.

* $$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$ = Matrix representation of the angular velocity vector.

* Dynamics of concentrated masses (particles)

Translation

$$\vec{F} = m\,\vec{a} \quad \text{for } \underline{\text{a single mass}}$$

$$F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k} = m\,(a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k})$$

For a <u>system of N particles</u> (i.e., masses)

$$\sum_{i=1}^{N} m_i\,\vec{a}_i = \sum_{i=1}^{N} \vec{F}_i$$

$\vec{F}_i$ = total external force acting on mass i

$$m\,\vec{a}_{CG} = \sum_{i=1}^{N} m_i\,\vec{a}_i$$

$\vec{a}_{CG}$ → acceleration of the centre of gravity/mass.

m ↓ Total mass $= \sum_{i=1}^{N} m_i$

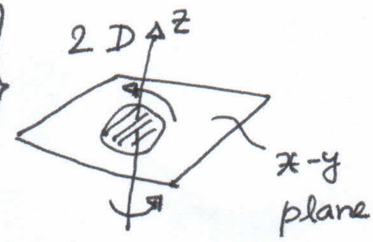
* <u>Rigid body in 2D</u>

$$\vec{T}_z = I_{zz}\,\vec{\alpha}_z$$

$\vec{\alpha}_z$ → angular acceleration

I_{zz} → Mass Moment of inertia

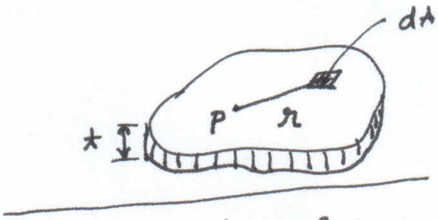
T_z ↑ Torque = moment about the z-axis

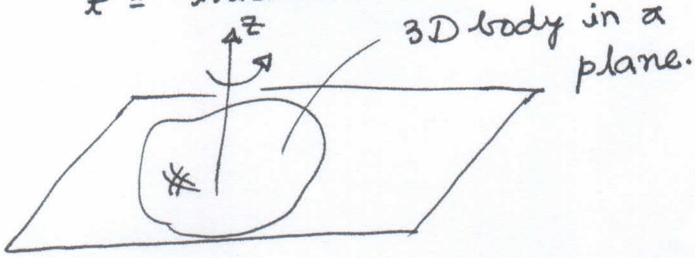


$$I_{zz} = \int_A \rho t\, r^2\, dA$$

ρ = mass per unit volume = mass density

t = thickness





$$I_{zz} = \int_{\text{volume}} \rho\, r^2\, dV$$

$\vec{T}_z$, $\vec{I}_{zz}$ should be about the same point, say, P.

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$$\underbrace{I_P}_{\text{Mass Moment of inertia about P}} = \underbrace{I_{CM}}_{\text{mass moment of inertia about the centre of mass}} + m\,d^2$$

d → distance between P and $\underbrace{\text{CM}}_{\text{centre of mass}}$.

m → mass

* Linear momentum

$\vec{p} = m\vec{v}$ for a single mass

$= \int \vec{v}\,\rho\,dV$ for a rigid body ($\vec{v} \to$ velocity)

$V \to$ Integration over the volume of a body.

$= m\,\vec{V}_{cm}$ $\quad$ $\vec{V}_{cm}$ = velocity of the centre of mass

$m \to$ mass of the body

* Angular momentum of a body = moment of momentum!

$$\vec{L}_p = \int_V \vec{r} \times (\vec{\omega} \times \vec{r})\,\rho\,dV$$

Position vector of a point from $P = \vec{r}$

$\vec{\omega}$ = angular velocity of the body $= (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$

ρ = mass density

$\vec{L}_p$ can be written as $\left\{ \begin{matrix} I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ -I_{yy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ -I_{zx}\omega_x + I_{xz}\omega_y + I_{zz}\omega_z \end{matrix} \right\}$

$$\Rightarrow \vec{L}_p = \underbrace{\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yy} & I_{zz} \end{bmatrix}}_{\text{mass Moment of inertia matrix}} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [\bar{\bar{I}}]\{\bar{\omega}\}$$

$I_{xx} = \int_V (y^2+z^2)\,\rho\,dV$ $\qquad$ $I_{xy} = \int_V xy\,\rho\,dv = I_{yx}$

$I_{yy} = \int_V (x^2+z^2)\,\rho\,dv$ $\qquad$ $I_{yz} = \int_V yz\,\rho\,dv = I_{zy}$

$I_{zz} = \int_V (x^2+y^2)\,\rho\,dv$ $\qquad$ $I_{zx} = \int_V zx\,\rho\,dv = I_{xz}$

$\bar{\bar{I}}$ is a tensor.

* Like stress and strain tensors, $\bar{\bar{I}}$ too will have principal planes where cross terms (i.e., the off-diagonal entries) in $\bar{\bar{I}}$ become zero.

They are called principal axes of inertia.

They can be found as follows.

$$\underbrace{\begin{bmatrix} I_{xx}-\lambda & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy}-\lambda & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz}-\lambda \end{bmatrix}}_{} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Equating the determinant of this gives a cubic in λ.

$\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\bar{\bar{I}}$.

Normalized Eigenvectors are the unit vectors are the axes.

$\lambda_1 = I_1$; $\lambda_2 = I_2$; $\lambda_3 = I_3$ } principal moments of inertia.

* Euler's eqns. of motion

$$\begin{cases} I_1 \alpha_1 - (I_2 - I_3)\omega_2\omega_3 = M_1 \\ I_2 \alpha_2 - (I_3 - I_1)\omega_1\omega_3 = M_2 \\ I_3 \alpha_3 - (I_1 - I_2)\omega_1\omega_2 = M_3 \end{cases}$$

1, 2, 3 refer to the principal axes of inertia.

If I's and Ms should be taken about a fixed point G, $\parallel$ $\vec{\alpha}_G, \vec{\omega}_G$ about some coordinate system,

$$[\bar{\bar{I}}_c]\{\vec{\alpha}\} + [\tilde{\omega}][\bar{\bar{I}}_c]\{\vec{\omega}\} = \{\vec{M}_c\}$$

* Kinetic energy = KE.

$KE = \frac{1}{2} m v^2$ for a particle

$= \underbrace{\frac{1}{2}\{\vec{\omega}\}[\bar{\bar{I}}]\{\vec{\omega}\}}_{\text{Rotational}} + \underbrace{\frac{1}{2} m \, \vec{V}_{cm} \cdot \vec{V}_{cm}}_{\text{Translational}}$ for a rigid body

* Potential energy = PE

PE = Strain energy + Work potential

$= SE + WP$

SE: = zero for a rigid body

WP: negative of the work done by non-dissipative external forces.

* $\mathcal{L}$ = Lagrangian = KE − PE

Equations of motion can be easily written using the Lagrangian.

Let $\mathcal{L}$ depend on N coordinates q_i $(i = 1, 2, \cdots N)$.

$\bar{q}$ comprise the dof of the body are the system.

<u>Equations of motion:</u>

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

$$\dot{q} = \frac{dq}{dt}$$

Any dissipative forces go there as they are not included in $\mathcal{L}$.

* Principal of minimum potential energy

This principle is an alternative way of ensuring static equilibrium just as the Lagrange's method of writing the equations of motion is an alternative to the Newton's law.

Here too, let PE be a function of N coordinates q_i = i^{th} dof $\quad i = 1, 2, \ldots, N.$

Then, minimum PE needs:

$$\frac{\partial PE}{\partial q_i} = 0 \qquad i = 1, 2, \ldots, N.$$

* Vibrations: oscillatory motion

(Special case of dynamics when there is mass (inertia) and stiffness; damping may also be there.)

A simple spring-mass system.

$$m\ddot{x} + kx = 0 \qquad \text{Free vibration.}$$

Natural frequency of oscillation $= \omega = \sqrt{\frac{k}{m}}$ rad/s

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz}$$

For a system of particles or a rigid body constrained by springs.

$$[M]\{\ddot{x}\} + [K]\{\vec{x}\} = \{\vec{0}\}$$

This system will have as many natural frequencies as the number of dof.