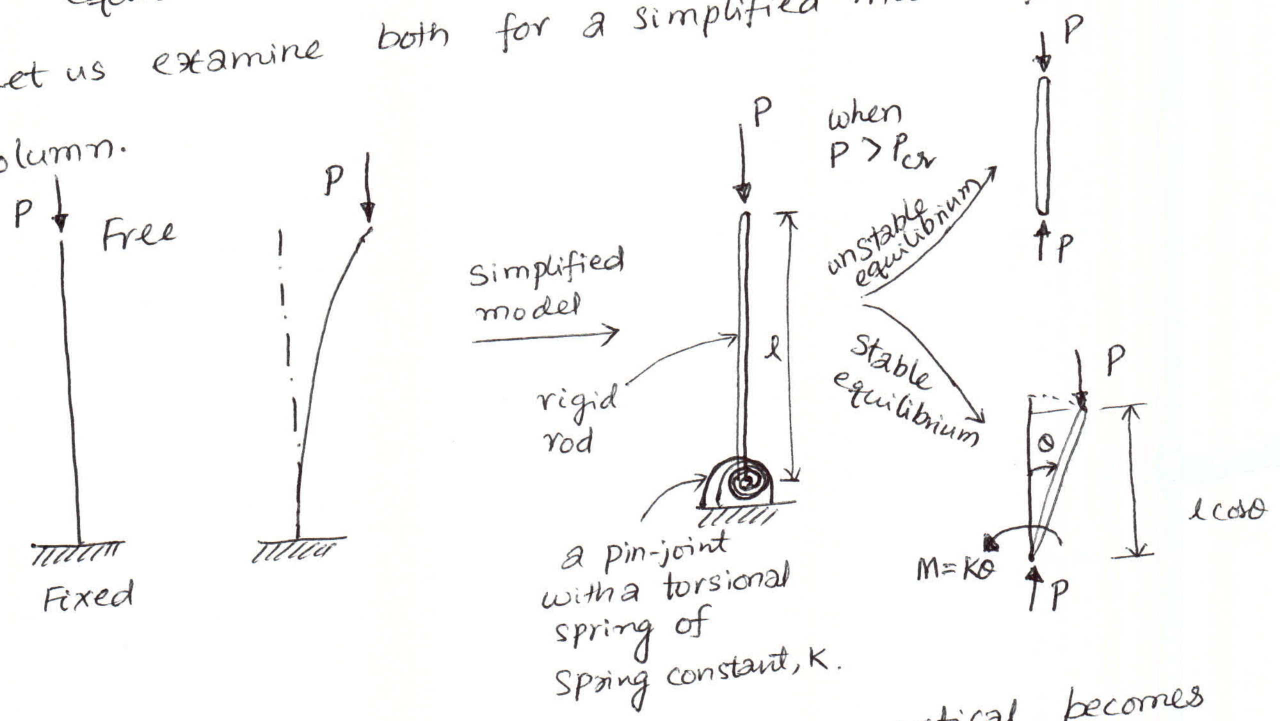
Buckling of columns

We all know that a wooden stick used to stir coffee bends when we hold it between two fingers and apply a compressive load along the length. This is called buckling. Why does a compressive load applied along the axis bend in the transverse direction? To understand the intuition behind buckling, we should first note that an elastic structure deforms upon application of a load to achieve static equilibrium. We have discussed two conditions for describing static equilibrium:

1) all forces should balance each other
2) potential energy should be a minimum for stable equilibrium (and a maximum for unstable equilibrium)

Let us examine both for a simplified model of a fixed-free column.



As shown, when $P > P_{cr}$, staying vertical becomes unstable and therefore the rod rotates to achieve stable equilibrium. But both satisfy static equilibrium conditions. In the rotated case, taking z-moment about the pin joint,

$$K\theta - Pl\sin\theta = 0 \Rightarrow \frac{\theta}{\sin\theta} = \frac{Pl}{K}.$$

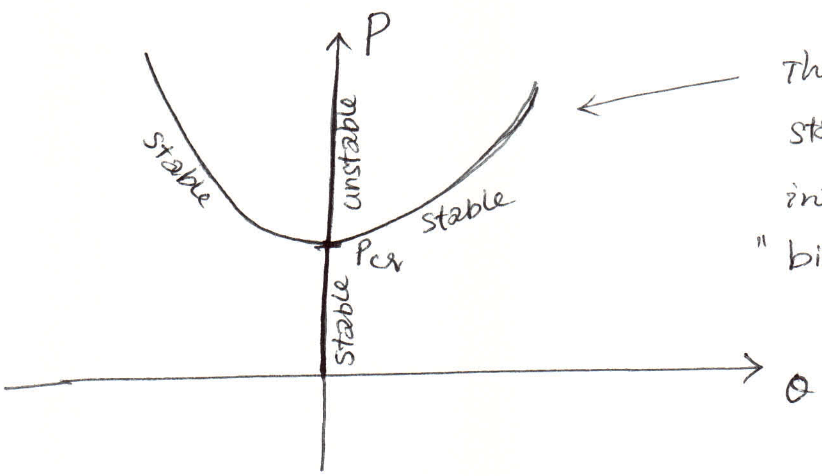
Thus, θ can be determined for a given value of P.

This equation has a solution if $\theta = 0$.

But it also has two more solutions when $P > P_{cr}$.

Solving for θ using $\frac{\theta}{\sin\theta} = \frac{Pl}{K}$ is difficult.

So, we will plot P for different values of θ in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then, we get:



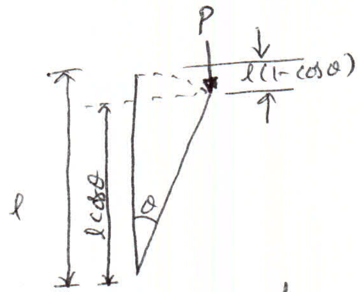
This phenomenon of a stable solution splitting into two solutions is called "bifurcation".

Note that, there is no reason for the rod to rotate only to the right. It can rotate by the same amount to the left. That is what the above plot tell us.

Unfortunately, force-balance method does not tell us why or how a static equilibrium solution is stable or unstable. Further analysis is required to study this using this method.

On the other hand, the principle of minimum potential energy makes it very clear as we see below.

Let us write potential energy (PE) as the sum of the strain energy (SE) stored in the torsional spring and the work potential (WP).



$$PE = SE + WP = \frac{1}{2}K\theta^2 - Pl(1-\cos\theta)$$

Necessary condition for a minimum is:

$$\frac{d(PE)}{d\theta} = 0 \Rightarrow K\theta - Pl\sin\theta = 0$$ (same as the one obtained with force balance).

$\theta = 0$ is always a solution.

To see if it is stable or unstable we should check the sufficient condition:

$\dfrac{d^2(PE)}{d\theta^2} > 0$ at a θ that satisfies the necessary condition.

$$\Rightarrow \frac{d^2(PE)}{d\theta^2} = K - Pl\cos\theta$$

At $\theta = 0$, $\dfrac{d^2(PE)}{d\theta^2} = K - Pl$.

$K - Pl > 0$ if $\dfrac{K}{l} > P$ or $P < \dfrac{K}{l}$.

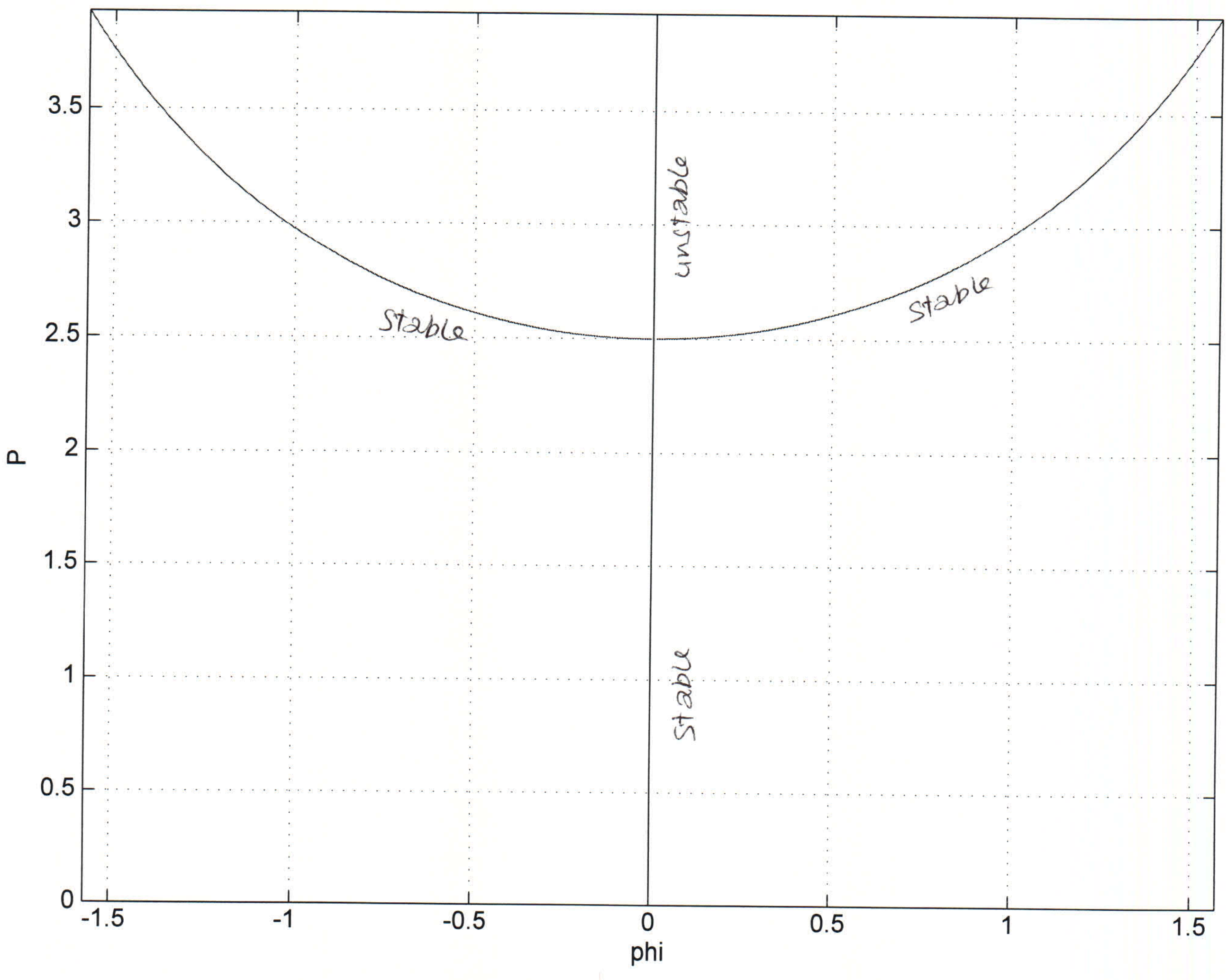
So, $\theta = 0$ is stable solution as along as P is less than $\dfrac{K}{l}$. After that $\theta = 0$ is unstable.

Thus, P_{cr} = critical buckling load $= \dfrac{K}{l}$ in this case.

Plots of PE in the subsequent pages make it clear how buckling occurs. Imagine a ball rolling on the PE curve!

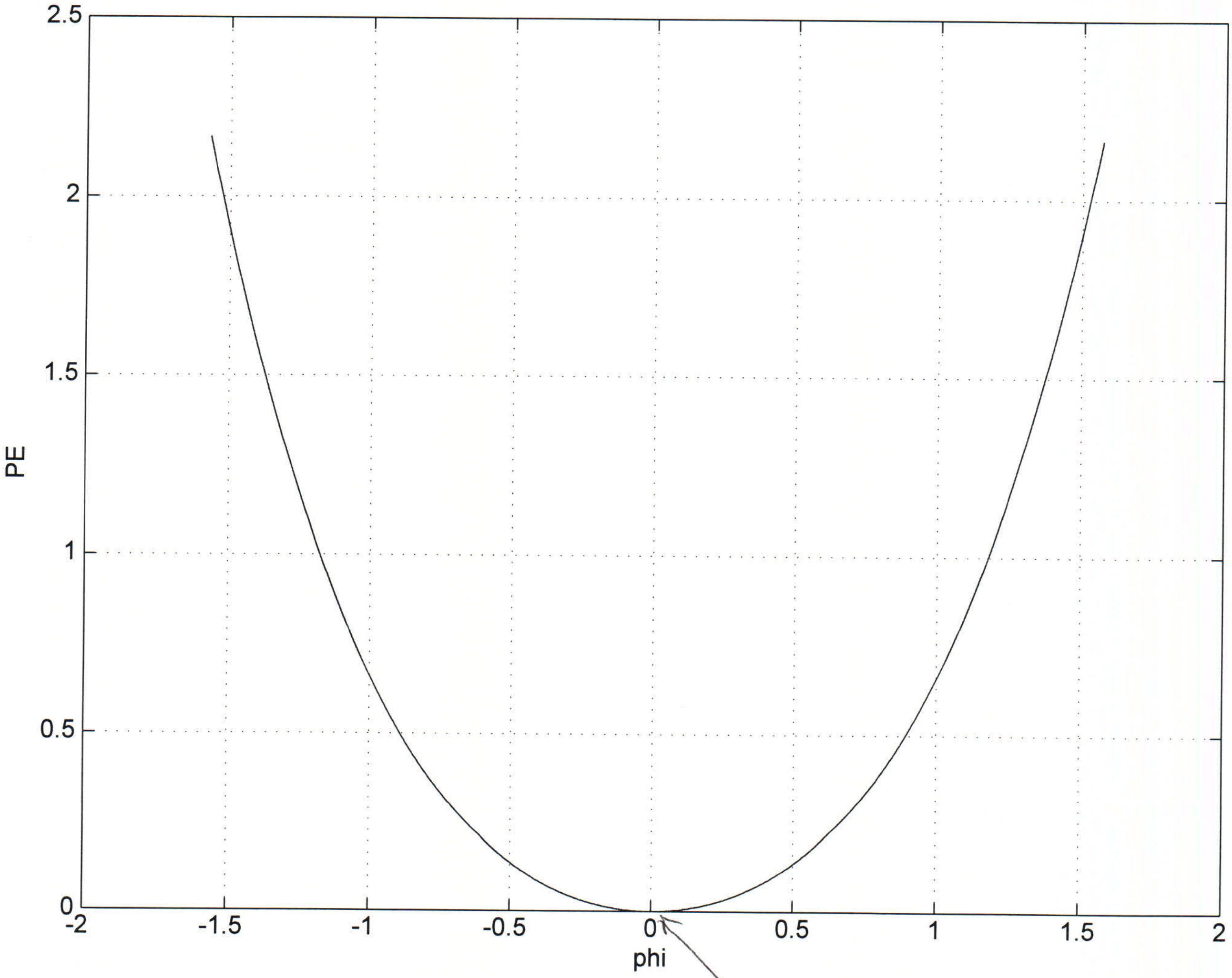
$K = 5$

$l = 2$



$$P = \frac{K\theta}{l \sin\theta}$$

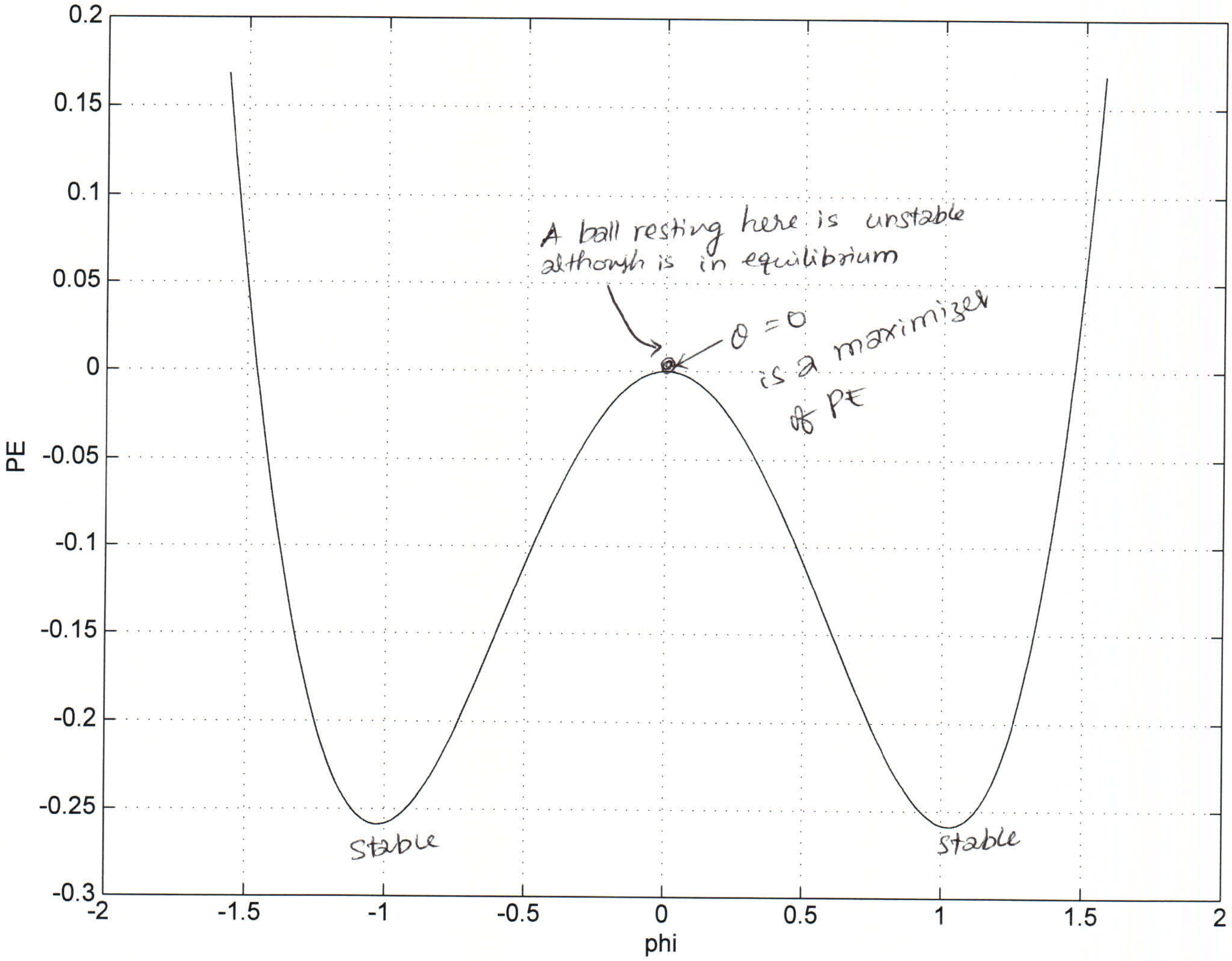
P is plotted for various values of θ.



$k = 5, \quad l = 2$

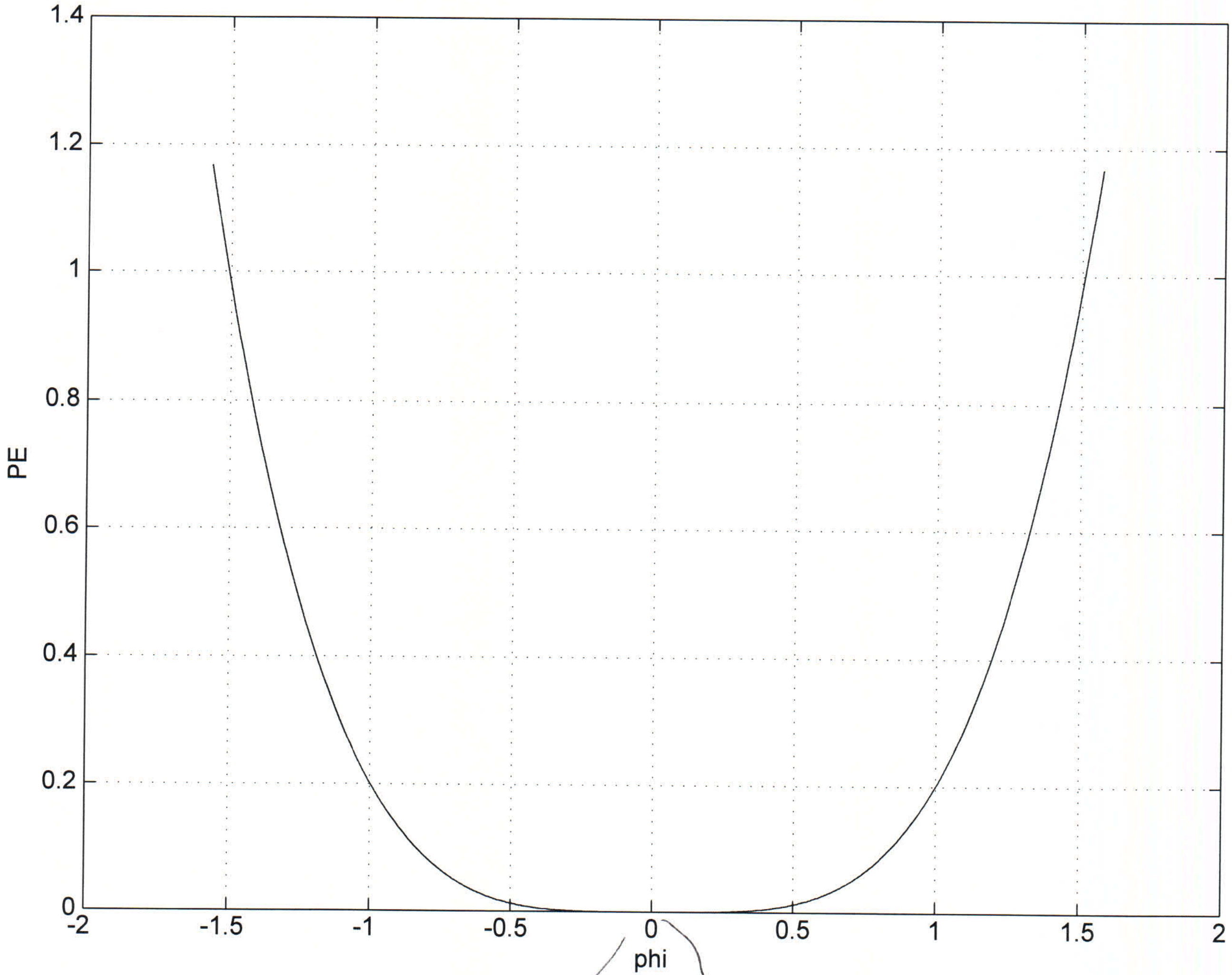
$P = 2 < \frac{k}{l}$

$\theta = 0$ is a minimizer of PE



$k = 5, \quad l = 2$

$P = 3 > \frac{k}{l}$



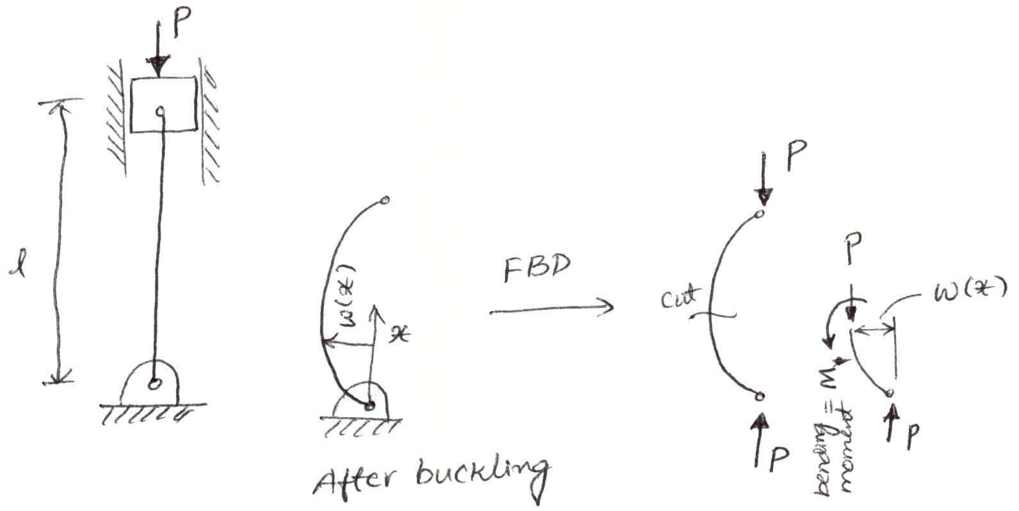
$P = 2.5 = \frac{K}{l}$

$\theta = 0$ is neutrally stable, i.e.
it is neither stable nor unstable.
This is when buckling occurs.

```
% Understand the buckling phenonmenon and bifurcation of solutions
clear all
clc
axis normal

k = 5;
l = 2;
Pcr = k/l;

phi = -pi/2:pi/360:pi/2;
Peq = k*phi./(l*sin(phi));
figure(1)
clf
plot(phi,Peq,'-r');
hold on
plot([0 0], [0 max(Peq)],'-b');
xlabel('phi');
ylabel('P');
axis([min(phi) max(phi) 0 max(Peq)]);
grid

P = 2.5;
PE = 0.5*k*phi.^2 - P*l*(1-cos(phi));

figure(2)
clf
plot(phi,PE,'-b');
xlabel('phi');
ylabel('PE');
grid
```

Having understood the phenomenon of buckling with a simplified model, let us return to the columns.

Case 1   Pinned-pined



After buckling

$$M = -Pw$$

$w(x) =$ transverse deflection of the buckled beam.

we know from beam-deflection formula that

$$M = -EI\frac{d^2w}{dx^2}$$

$$\therefore \quad M = -EI\frac{d^2w}{dx^2} = -Pw$$

$$\Rightarrow \quad EI\,w'' + Pw = 0 \qquad \left(w'' = \frac{d^2w}{dx^2}\right)$$

A general solution to this equation is:

$$w = A\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + B\sin\left(\sqrt{\frac{P}{EI}}\,x\right)$$

Boundary conditions are: $w=0$ @ $x=0$

$w=0$ @ $x=l$

$w = 0$ @ $x = 0$ $\Rightarrow$ $A = 0$

$w = 0$ @ $x = l$ then implies: $\cancel{A}^{0} \cos\left(\sqrt{\frac{P}{EI}}\, x\right) + B \sin\left(\sqrt{\frac{P}{EI}}\, l\right) = 0$

$$\Rightarrow \quad B \sin\left(\sqrt{\frac{P}{EI}}\, l\right) = 0$$

$B = 0$ is a trivial solution (i.e. $w = 0$ for all x. the column stays vertical, which as we know is an unstable solution when $P > P_{cr}$).

Non-trivial solution ($B \neq 0$) is:

$$\sin\left(\sqrt{\frac{P}{EI}}\, l\right) = 0 \Rightarrow \sqrt{\frac{P}{EI}}\, l = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad P = \frac{n^2 \pi^2 EI}{l^2} = P_{cr}$$

For $n = 1$,

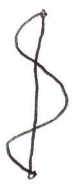
$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

For $n = 2$,

$$P_{cr} = \frac{4\pi^2 EI}{l^2}$$

For $n = 3$,

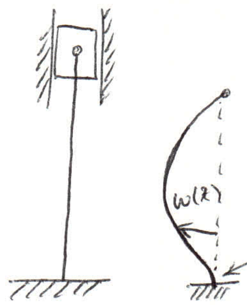
$$P_{cr} = \frac{9\pi^2 EI}{l^2}$$



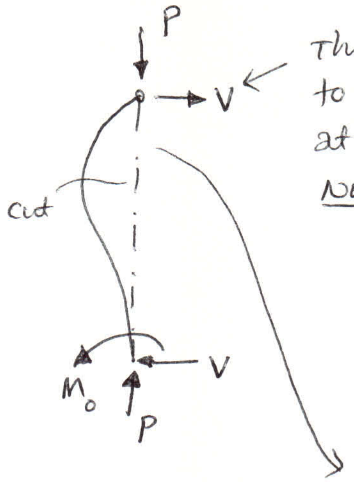
and so on for $n = 4, 5, \ldots$

Case 2

Fixed - pinned

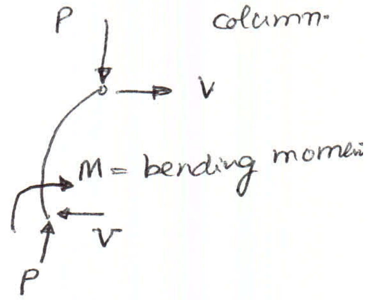


must have zero slope because of fixed condition



This is necessary to balance M_0 acting at the other end.

Note $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_z = 0$ should be satisfied for the Buckled column.



Note $M = -EIw''$

$$-M - V(l-x) - Pw = 0$$

$$\Rightarrow \quad M = -Pw - V(l-x)$$

$$-EIw'' - Pw - V(l-x) = 0$$

$$\Rightarrow \quad Pw + EIw'' = V(x-l) \Rightarrow \quad w'' + \frac{P}{EI} w = \frac{V}{EI}(x-l)$$

Solution of this equation is :

$$w = A\cos\left(\sqrt{\frac{P}{EI}}\, x\right) + B\sin\left(\sqrt{\frac{P}{EI}}\, x\right) + \frac{V}{P}(x-l)$$

Boundary conditions are :

$w = 0$ @ $x = 0$

$w' = 0$ @ $x = 0$

$w = 0$ @ $x = l$

$$w=0 \text{ @ } x=0 \quad \Rightarrow \quad A - \frac{Vl}{P} = 0 \quad \Rightarrow \quad A = \frac{Vl}{P}.$$

$$w'=0 \text{ @ } x=0 \quad \Rightarrow \quad B\sqrt{\frac{P}{EI}} + \frac{V}{P} = 0 \quad \Rightarrow \quad B = -\frac{V}{P}\sqrt{\frac{EI}{P}}.$$

$w = 0$ @ $x = l$ $\Rightarrow$ $$A\cos\left(\sqrt{\frac{P}{EI}}\,l\right) + B\sin\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$$

$$\Rightarrow \quad \tan\left(\sqrt{\frac{P}{EI}}\,l\right) = -\frac{A}{B}$$

But from previous two boundary conditions, we have, $A = \frac{Vl}{P}$ and $B = -\frac{V}{P}\sqrt{\frac{EI}{P}}$

$$\therefore \quad \tan\left(\sqrt{\frac{P}{EI}}\,l\right) = -\frac{A}{B} = -\frac{\cancel{V}l}{\cancel{P}} \times \frac{\cancel{P}\sqrt{P}}{-\cancel{V}\sqrt{EI}} = \sqrt{\frac{P}{EI}}\,l$$

Let $\sqrt{\frac{P}{EI}}\,l = \alpha$; we have to solve for $\tan\alpha = \alpha$.

$\alpha = 4.4934$ is a solution. (plot $y = \tan\alpha$ and $y = \alpha$ and find their intersection point.)

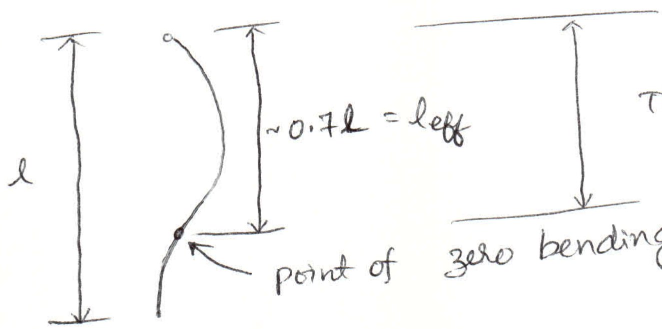
$$\Rightarrow \quad \sqrt{\frac{P}{EI}}\,l = 4.4934$$

$$\Rightarrow \quad P = P_{cr} = \frac{20.1906\,EI}{l^2} = \frac{\pi^2 EI}{l_{eff}^2}$$

to cast it in the form of pinned-pinned formula.

where $l_{eff} \approx 0.7\,l$.

Intuition for l_{eff}

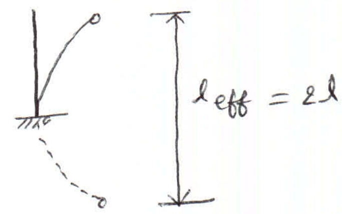


This portion is equivalent to the pinned-pinned case.

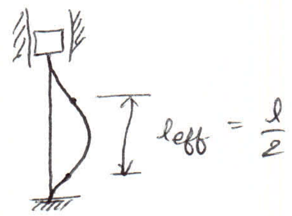
Now, we don't need to solve for other cases because we can easily define l_{eff}.

→ of course, this requires work, but someone has already done that.

<u>Fixed-free</u> $\quad l_{eff} = 2l$ →



<u>Fixed-fixed</u> $\quad l_{eff} = \frac{l}{2}$ →



We consider that a column has failed if it buckles or if it gets crushed (or yielding) under compressive loading when $|\sigma_c|$ exceeds S_y. A short column fails by getting crushed (or yielding) and a long column fails by buckling. We define a slenderness ratio to characterize a column as short or long.

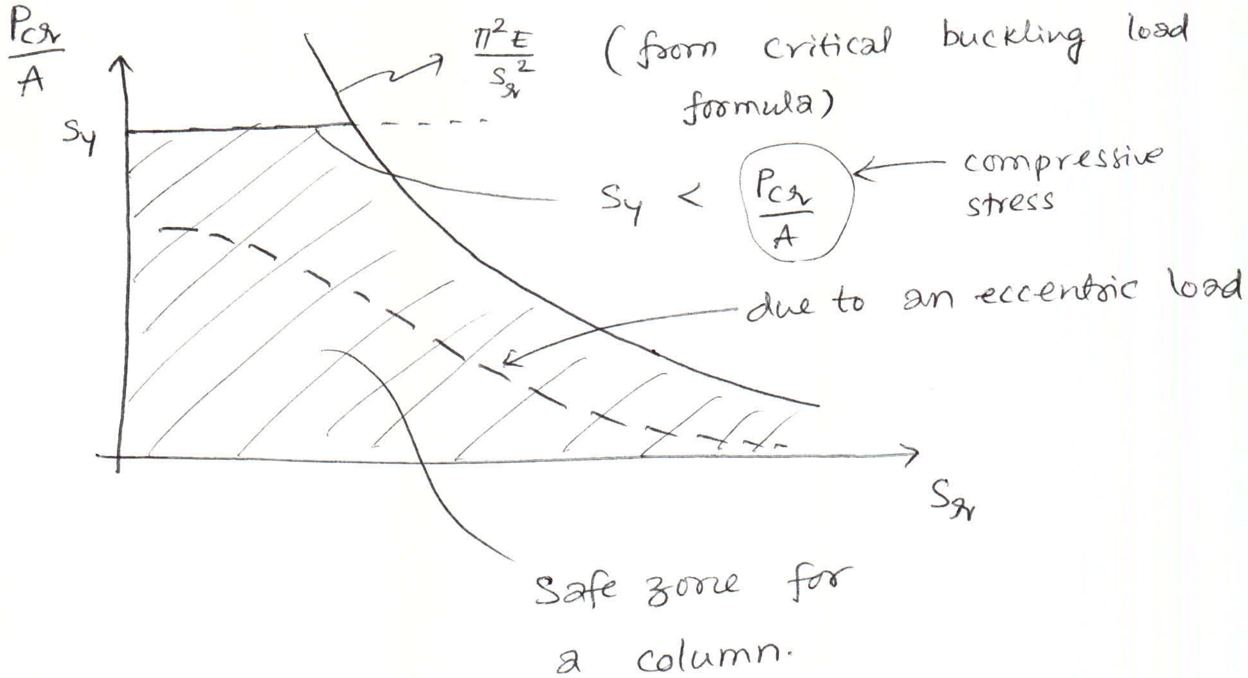
$$S_r = \text{slenderness ratio} = \frac{l_{eff}}{K}$$

where K = radius of gyration $= \sqrt{\frac{I}{A}}$

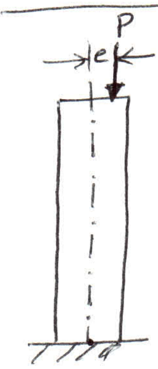
I → moment of inertia of c/s

A → area of c/s.

Now,
$$\frac{P_{cr}}{A} = \frac{\pi^2 E \cancel{A} K^2}{l_{eff}^2} = \frac{\pi^2 E}{S_r^2}$$



* Eccentric columns



If the load is eccentric, i.e. if it acts slightly off from the neutral axis, the critical load reduces because the moment due to the eccentric load tends to bend the column for a much smaller P.

Then,
$$P_{cr} = A\left\{\frac{S_y}{1 + E_r \sec\left(\frac{l_{eff}}{K}\sqrt{\frac{P_{cr}}{4EA}}\right)}\right\}$$

where $E_r = \frac{ec}{K^2}$ where c = maximum distance of a point in the cross-section from the neutral axis.

$\frac{P_{cr}}{A}$ under eccentric loading is also shown in the plot above.