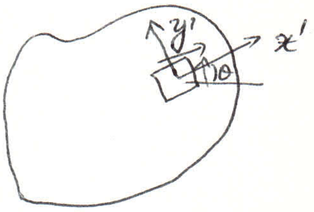
Coordinate transformations of stresses and principal stresses

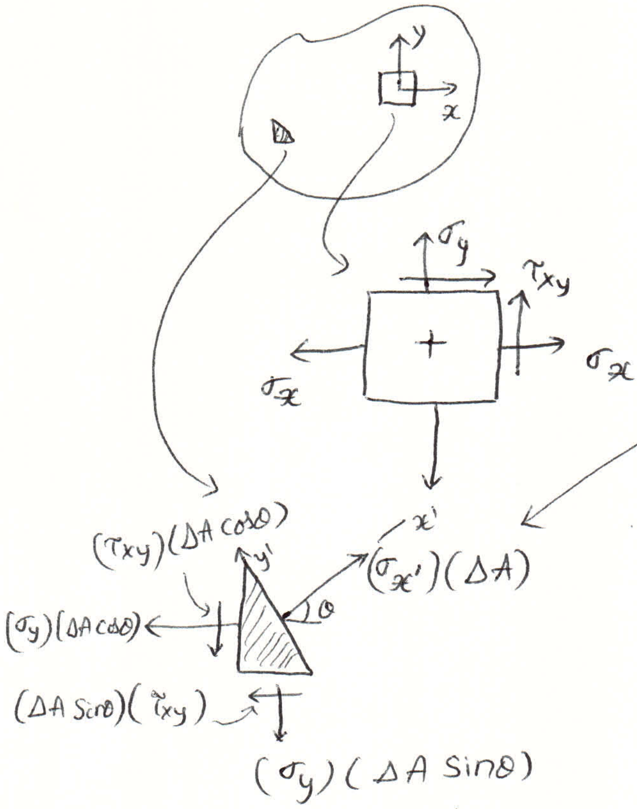
As you know, choice of a coordinate system is arbitrary. So, if you obtained σ_x, σ_y, τ_{xy}, ... in your coordinate system at a point, someone else might have obtained different values for a different coordinate system. How do we compare them? we need to transform stresses in one coordinate system to another. It is easier than you think it is... as always, we use FBDs.



$\sigma_{x'} = ?$

$\sigma_{y'} = ?$

$\tau_{x'y'} = ?$



For this oblique differential element,

$$\sum F_{x'} = 0$$

$$\Rightarrow \sigma_{x'} \Delta A - \sigma_x (\Delta A \cos\theta)\cos\theta - \tau_{xy}(\Delta A \cos\theta)\sin\theta - \sigma_y(\Delta A \sin\theta)\sin\theta - \tau_{xy}(\Delta A \sin\theta)\cos\theta = 0$$

$$\Rightarrow \sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

Re-arrange to get

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{(1)}$$

$\Sigma F_{y'} = 0$ gives

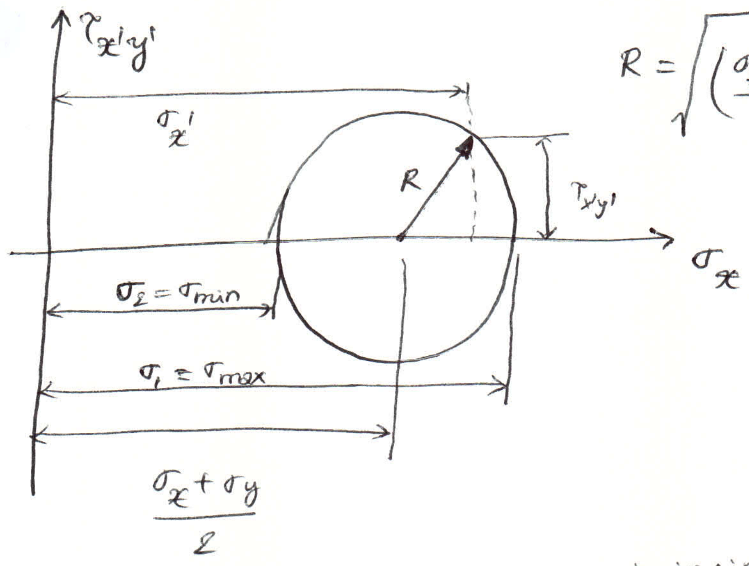
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \longrightarrow (2)$$

By taking an oblique element with face along y' axis, we can get

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \longrightarrow (3)$$

Equns. (1) - (3) should remind of Mohr's circle. Check it out from the following figure.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



$\sigma_1 = \sigma_{max}$ and $\sigma_2 = \sigma_{min}$ are principal stresses which occur at an angle θ_p given by

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}.$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

max. shear stress $= \tau_{21} = \frac{\sigma_1 - \sigma_2}{2}$

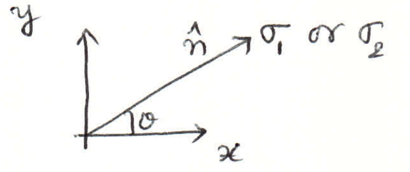
occurs when $\sigma_{x'} = \sigma_1$ and $\sigma_{y'} = \sigma_2$.

The method we just used is nice and convenient but becomes tedius if we attempt to transform 3-D stresses: σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}. So, let us develop a method for 2-D that we can use for 3-D as well just as easily.

Consider

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \sigma \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$$

where n_x and n_y are the direction cosines of the normal. ~~to~~ along the σ_1 (max. stress) axis or σ_2 (min. stress) axis. In 2-D, $n_x = \cos\theta$; $n_y = \sin\theta$



n_x = cosine of angle made by unit vector $\hat{n}$ with x-axis $= \cos\theta$

n_y = cosine of angle made by $\hat{n}$ with y-axis $= \cos(90° - \theta) = \sin\theta$

Re-write as

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This has a non-trivial solution only if the determinant of the matrix is zero.

$$\det(\text{matrix}) = \det\left(\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{bmatrix}\right)$$

$$= (\sigma_x - \sigma)(\sigma_y - \sigma) - \tau_{xy}^2 = 0$$

$$\Rightarrow \quad \sigma_x \sigma_y - \sigma(\sigma_x + \sigma_y) + \sigma^2 - \tau_{xy}^2 = 0$$

$$\Rightarrow \quad \sigma^2 - \sigma(\sigma_x + \sigma_y) + \sigma_x \sigma_y - \tau_{xy}^2 = 0$$

$$\Rightarrow \quad \sigma = \frac{(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x + \sigma_y)^2 - 4(\sigma_x \sigma_y - \tau_{xy})^2}}{2}$$

$$\Rightarrow \quad \sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Thus, the two values of σ turned out to be σ_1 & σ_2!

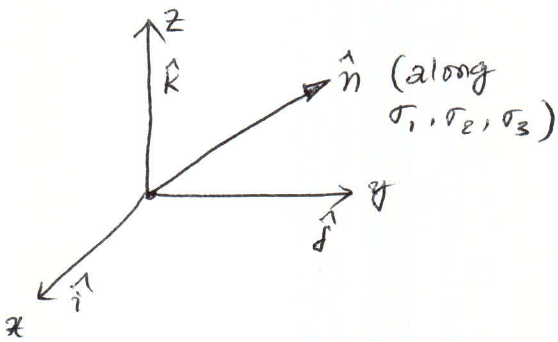
If you now solve for n_x ~~and~~ or n_y, you can θ as well. Let us call it θ_p to be consistent with our notation.

<u>Exercise</u> verify θ_p expression.

How do we go to 3-D?

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} n_y \\ n_y \\ n_z \end{Bmatrix} = \sigma \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

$$\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$$



$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Again, for a non-trivial solution, determinant of the matrix must be zero.

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

$$\Rightarrow \quad \sigma^3 - C_2 \sigma^2 - C_1 \sigma - C_0 = 0$$

Called characteristic (stress cubic) equn.

where
$$C_2 = \sigma_x + \sigma_y + \sigma_z$$
$$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$$
$$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

c_0, c_1, and c_2 are co-ordinate system invariants. That is, their values do not change if we compute them in any coordinate system.

You can easily verify it in 2-D. Go back and look for c_0 and c_1.

$\boxed{}$ ← write 2-D characteristic (stress quadratic) equn. here.

compare with

$\sigma^2 - c_2\sigma^2 - c_1 = 0$

$\Rightarrow \quad c_2 = \boxed{}$

$c_1 = \boxed{}$

$\sigma_1, \sigma_2, \sigma_3$ principal stresses in 3-D are solutions of the cubic equn in the previous page.

we order them as: $\sigma_1 > \sigma_2 > \sigma_3$

Maximum shear stresses are:

$\tau_{13} = \dfrac{\sigma_1 - \sigma_3}{2}$; $\tau_{21} = \dfrac{\sigma_1 - \sigma_2}{2}$;

$\tau_{32} = \dfrac{\sigma_2 - \sigma_3}{2}$.