

why is there a π in the Green's function solution of the diffusion equation?

$$p(x, t) = \frac{c(x, t)}{N}$$

N particles were kept at $x=0$ at $t=0$.

$$\int_{-\infty}^{\infty} p(x, t) dx = 1.$$

What if $p(x, t) = \frac{1}{2\sqrt{Dt}} e^{-x^2/4Dt}$? (without π !)

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{Dt}} e^{-x^2/4Dt} dx = \frac{1}{2\sqrt{Dt}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}} dx = \frac{1}{2\sqrt{Dt}} \cdot \frac{\sqrt{\pi}}{(2\sqrt{Dt})} = \frac{\sqrt{\pi}}{4Dt}$$

So, there must be π .

Note: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$

How? consider $\left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\alpha y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx dy$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-\alpha r^2} r dr d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\infty} e^{-\alpha r^2} r dr \right)$$

$$\int_0^{\infty} e^{-u} \frac{du}{(-2\alpha)} = -\frac{1}{2\alpha} e^{-u} \Big|_0^{\infty} \quad \text{let } u = -\alpha r^2$$

$$du = -2\alpha r dr \Rightarrow r dr = \frac{du}{-2\alpha}$$

$$= \frac{1}{2\alpha}$$

$$\therefore \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{2\pi}{\alpha}} = \sqrt{\frac{\pi}{\alpha}}$$