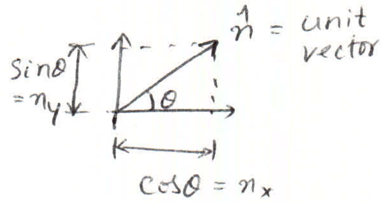
Calculating principal stresses – an alternate method

Note that, using force-balance method on an oblique differentially small element, we got normal stress $\sigma_{x'}$ along a normal $\hat{n}$.

$$\hat{n} = n_x \hat{i} + n_y \hat{j} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}$$



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Let us play a trigonometric trick on it to get

$$\sigma_{x'} = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) + \tau_{xy}\sin 2\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= \sigma_x n_x^2 + \sigma_y n_y^2 + 2\tau_{xy} n_x n_y$$

Let us use matrix multiplication to render it in a different form.

$$\sigma_{x'} = \{n_x \quad n_y\} \begin{Bmatrix} \sigma_x n_x + \tau_{xy} n_y \\ \sigma_y n_y + \tau_{xy} n_x \end{Bmatrix}$$

$$\Rightarrow \sigma_{x'} = \{n_x \quad n_y\} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \longrightarrow (1)$$

$$= \hat{n}^T \cdot \underset{\sim}{S} \cdot \hat{n}$$

where $\hat{n} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$; $\underbrace{\hat{n}^T}_{\text{Transpose}} = \{n_x \quad n_y\}$

$\underset{\sim}{S} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$ = stress matrix

Keep this in mind as it extends naturally to 3-D stress-state also.

Note also that $n_x^2 + n_y^2 = \cos^2\theta + \sin^2\theta = 1 \longrightarrow$ (2)

We want to find n_x & n_y such that $\sigma_{x'}$ becomes largest or smallest because that is the definition of principal stresses. But then n_x and n_y must satisfy (2). So, it is a constrained optimization (minimization or maximization) problem.

optimize $\sigma_{x'}$
Subject to $n_x^2 + n_y^2 - 1 = 0$
by varying n_x and n_y.

Called a constraint

$\Rightarrow$

optimize $\underbrace{\sigma_{x'} - \lambda(n_x^2+n_y^2-1)}_{\mathcal{L} = \text{Lagrang}}$
by varying n_x, n_y, and λ.
λ = Lagrange multiplier

optimize $\mathcal{L} = \hat{n}^T S \hat{n} - \lambda(\hat{n}^T\hat{n} - 1)$
by varying $\hat{n}$ and λ

Necessary conditions for $\mathcal{L}$ (= Lagrangian) to be a minimum or a maximum are:

$$\frac{\partial \mathcal{L}}{\partial \hat{n}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

Note $\hat{n}^T\hat{n} = n_x^2 + n_y^2$

$$S\hat{n} - \lambda\hat{n} = 0 \Rightarrow \underbrace{\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \lambda \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}}$$

& $\underbrace{\hat{n}^T\hat{n} - 1 = 0}$

we got back the constraint!

This is exactly what we have in the other handout.

Solve it to get λ. You will get two values and they are principal stresses.