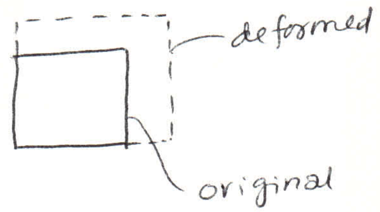
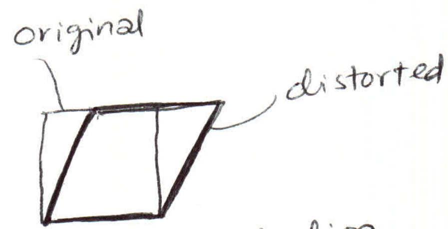
Static Failure Criteria - Part II

Distortion energy criterion

When an elastic object deforms, it develops strain energy. This strain energy consists of two parts: one that changes the volume of the object and the other that distorts the object. The part that changes the volume is called <u>hydrostatic strain energy</u> and the other is called <u>distortion energy</u>.



pure volume change
(only hydrostatic strain energy)



pure distortion
(only distortion energy)

stores both hydrostatic strain energy and distortion energy

in general

$$\underbrace{SE}_{\text{strain energy}} = \underbrace{SE_h}_{\text{hydrostatic strain energy}} + \underbrace{DE}_{\text{distortion energy}}$$

Microscopic yielding of material is due to relative sliding of atoms, which distorts the object at the macroscopic level. So, distortion energy is believed to be a measure of how close an object is to failure. The hypothesis for this criterion can be stated as follows.

Hypothesis

"When the distortion energy at a critical point in the object equals the distortion energy at the critical point in a failed tensile test specimen, we say that the object also fails by yielding."

(Recall that failure for ductile materials is yielding for our purposes.)

Let us now discuss how to compute the distortion energy. We begin with strain energy.

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se = strain energy per unit volume = $\frac{1}{2}\sigma^T \varepsilon$

= <u>strain energy density</u>

In vector notation,

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \quad \text{and} \quad \varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\Rightarrow se = \frac{1}{2}\{\sigma_x \;\; \sigma_y \;\; \sigma_z \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{zx}\}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$= \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

Or in terms of principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ and strains on principal stress planes $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, we get

(<u>Note</u> principal stress planes have zero shear stresses).

$$se = \frac{1}{2}(\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3).$$

From Hookes law,

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3)$$

$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

ν = Poisson's ratio
E = Young's modulus

Then, se becomes

$$se = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right)$$

Just as the strain energy has two components namely hydrostatic and distortion, the stress can also be split into similar components. Doing this is useful to us because we got se in terms of stresses only above.

stress = hydrostatic stress + $\underbrace{\text{distortion}}_{}$ stress
also called "deviatoric"

$\therefore$

$$\sigma_1 = \sigma_h + \sigma_{1d}$$

$$\sigma_2 = \sigma_h + \sigma_{2d}$$

$$\sigma_3 = \sigma_h + \sigma_{3d}$$

Note, by definition, hydrostatic component of stress is the same in all directions.

How do we find σ_h?

Let us add them up.

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h + (\sigma_{1d} + \sigma_{2d} + \sigma_{3d})$$

$$\Rightarrow \sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3 - (\sigma_{1d} + \sigma_{2d} + \sigma_{3d})}{3}.$$

But we do not know σ_{1d}, σ_{2d}, and σ_{3d}. We do know that they don't cause change in volume because they only distort the object. That gives us an idea to compute them. To calculate volume change, we want to think in terms of strains.

$$\left.\begin{aligned}
\varepsilon_{1d} &= \frac{1}{E}\left(\sigma_{1d} - \nu\sigma_{2d} - \nu\sigma_{3d}\right)\\
\varepsilon_{2d} &= \frac{1}{E}\left(\sigma_{2d} - \nu\sigma_{1d} - \nu\sigma_{3d}\right)\\
\varepsilon_{3d} &= \frac{1}{E}\left(\sigma_{3d} - \nu\sigma_{1d} - \nu\sigma_{2d}\right)
\end{aligned}\right\} \text{ Hooke's law}$$

Adding these up, we get

$$\underbrace{\varepsilon_{1d} + \varepsilon_{2d} + \varepsilon_{3d}}_{} = \frac{1-2\nu}{E}\left(\sigma_{1d} + \sigma_{2d} + \sigma_{3d}\right)$$

This part is proportional to volume-change and must be zero.

$\therefore \quad \sigma_{1d} + \sigma_{2d} + \sigma_{3d} = 0$ as well.

$$\therefore \quad \sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3 - \cancelto{0}{(\sigma_{1d} + \sigma_{2d} + \sigma_{3d})}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}.$$

To see why this is volume change, consider a cube of dimensions a, b, and c which have changed to $a+da$, $b+db$, and $c+dc$ upon deformation of the cube.

$$\text{change in volume} = (a+da)(b+db)(c+dc) - abc$$

$$= \cancel{abc} + ab\,dc + ac\,db + bc\,da + \underbrace{\cancelto{0}{a\,db\,dc} + \cancelto{0}{b\,da\,dc} + \cancelto{0}{c\,da\,db}}_{\text{second order terms}} + \underbrace{\cancelto{0}{da\,db\,dc}}_{\text{third order term}}$$

$$- \cancel{abc}$$

$$= abc\left(\underbrace{\frac{dc}{c}}_{\varepsilon_3} + \underbrace{\frac{db}{b}}_{\varepsilon_2} + \underbrace{\frac{da}{a}}_{\varepsilon_1}\right) = \text{volume}_{\text{original}}\left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3\right).$$

Recall that

$$se = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right)$$

Since, $\sigma_{1h} = \sigma_{2h} = \sigma_{3h} = \sigma_h$,

se_h = hydrostatic strain energy density

$$= \frac{1}{2E}\left(\sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu\left(\sigma_h^2 + \sigma_h^2 + \sigma_h^2\right)\right)$$

$$= \frac{3}{2}\left(\frac{1-2\nu}{E}\right)\sigma_h^2 = \frac{3}{2}\left(\frac{1-2\nu}{E}\right)\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2$$

Since $se = se_h + de$

de: distortion energy density = distortion energy per unit volume

$$de = se - se_h$$

$$= \frac{1}{2E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right\} - \frac{3}{2}\left(\frac{1-2\nu}{E}\right)\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2$$

$$= \frac{1+\nu}{3E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right\}$$

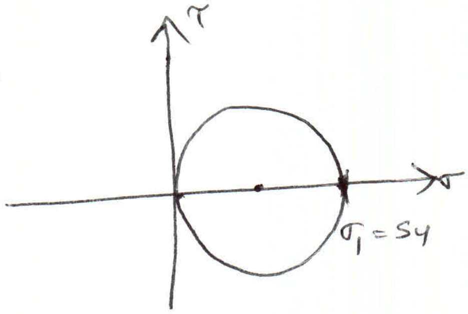
Thus, in terms of principal stresses, de is

$$\boxed{de = \frac{1+\nu}{3E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right\}}$$

Recall our hypothesis for this criterion (page 2).

de in the tensile test specimen = de in the object

when yielding occurs.



<u>de in the tensile test specimen</u>

$\sigma_1 = S_y$; $\sigma_2 = 0$; $\sigma_3 = 0$

$$\therefore \quad de = \frac{1+\nu}{3E}\left(S_y^2\right)$$

According to our hypothesis, if the most critical point in our object has principal stresses σ_1, σ_2, and σ_3,

then,

$$\underbrace{\frac{1+\nu}{3E} S_y^2}_{\text{de in the specimen}} = \frac{1+\nu}{3E}\underbrace{\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right)}_{\text{de in the object}}$$

$$\Rightarrow \quad S_y^2 = \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right)$$

$$\text{or} \quad S_y = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

$= \sigma'$ = von Mises stress $\Leftarrow$

Since principal stresses, σ_1, σ_2, and σ_3, are co-ordinate system invariant, so is Von Mises stress, σ'!

Thus, in this criterion, we have obtained a single stress (called von Mises stress) that includes all three principal stresses, and in turn, all six stresses (σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}) for a chosen coordinate system.

Let us state the failure criterion.

"when von Mises stress equals S_y, yielding will occur".

To avoid yielding, $\sigma' \leq S_y$

To show this theory graphically in 2-D, let us substitute $\sigma_2' = 0$.

Then, $\sigma' = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3}$

$$\sigma' < S_y \Rightarrow \sigma'^2 \leq S_y^2$$

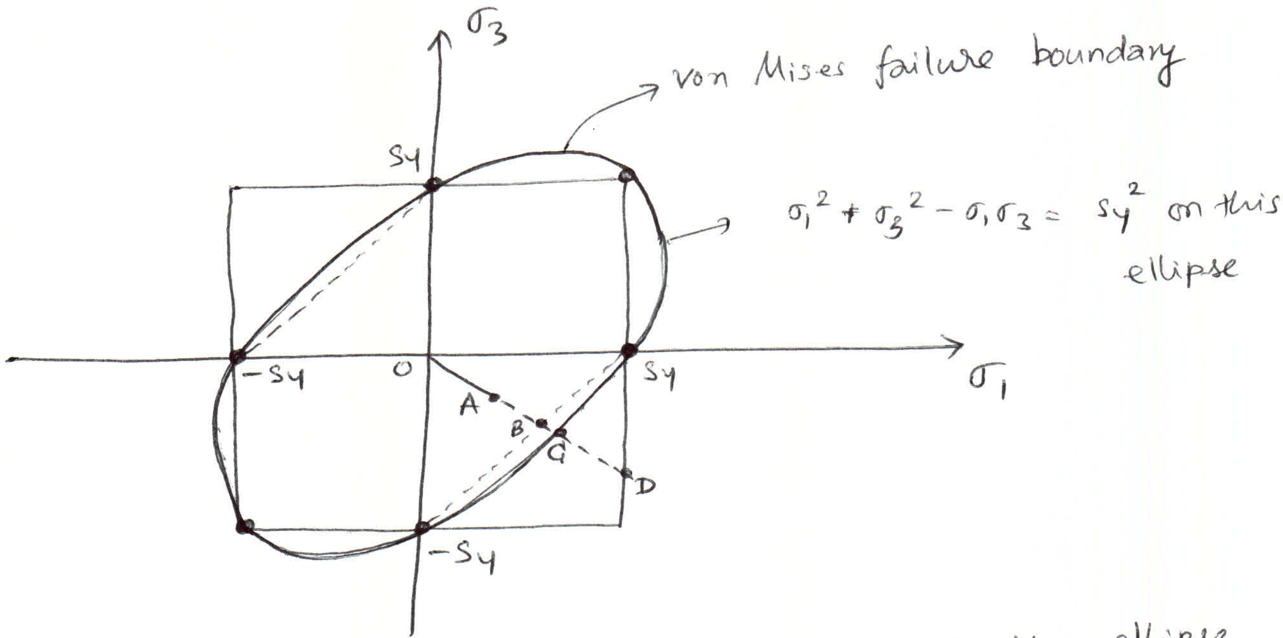
$$\Rightarrow \sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 \leq S_y^2$$

In the limiting case (to get failure boundary),

$$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = S_y^2$$

This is the equation of an inclined ellipse in the $\sigma_1 - \sigma_3$ plane.

To see this, let us plot it.



Safe zone under this criterion is <u>inside the ellipse</u>.

Comparing the three criteria we have discussed, we see:

max. shear stress (inclined hexagon) is the <u>most conservative</u> and gives smallest factor of safety.

FS: factor of safety

$$= \frac{OB}{OA} \quad \text{(max. shear stress criterion)}$$

$$= \frac{OC}{OA} \quad \text{(von Mises distortion energy criterion)}$$

$$= \frac{OD}{OA} \quad \text{(max. normal stress criterion)}$$

clearly, $OB < OC < OD$.

In max. shear stress and max. normal stress criteria, we need to check which quadrant we are in to compute the factor of safety in our calculations. In von Mises criterion, it is easy and straight forward no matter which quadrant we are in.

$$\boxed{\begin{array}{l} \text{Factor of safety} = FS = \dfrac{S_y}{\sigma'} \\ \text{in Von Mises criterion} \\[1em] \text{In 2D, } \sigma' = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} \\[1em] \text{2D, } \sigma' = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \end{array}}$$

Exercise

Express σ' in terms of $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ directly instead of principal stresses, σ_1, σ_2, and σ_3.

Ans.

$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

In 2-D,

$$\sigma' = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2}$$

Hint: The hard way would be to compute σ_1, σ_2, and σ_3 in terms of $\sigma_x, \sigma_y, \ldots$ etc. and substitute in σ' expression. Easy way is to use "invariants" in the "stress cubic" we had discussed when we talked about principal stresses.