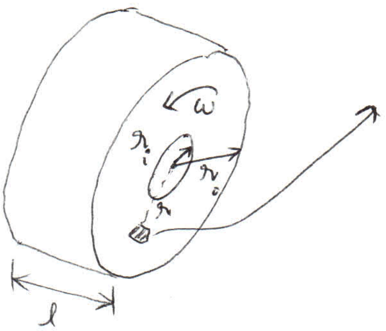
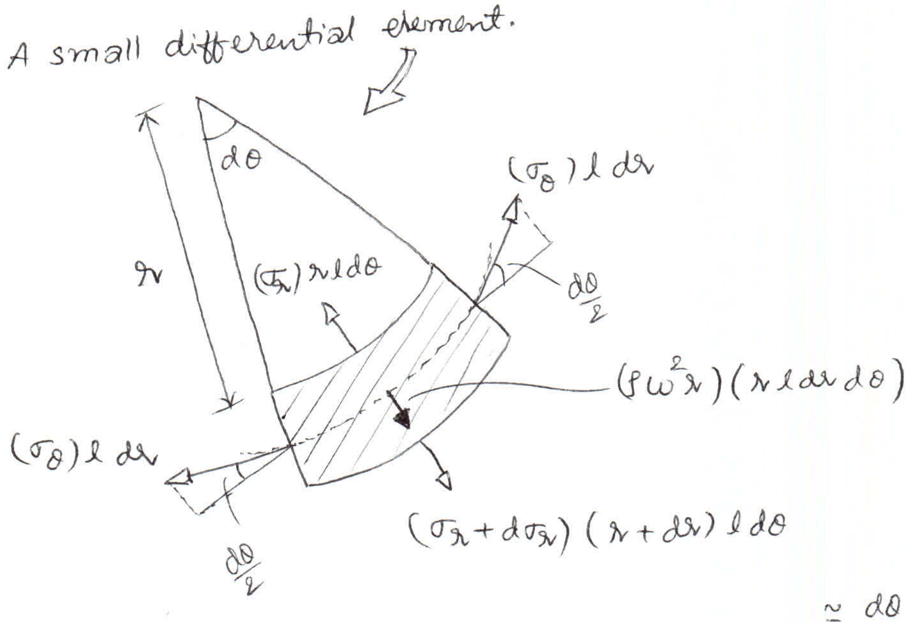
Thick-walled cylinders Revisited - Rotating disks

Previously, we had analyzed thick-walled cylinders as pressure vessels. Now, we consider a rotating disk such as a flywheel. In pressure vessels, we had internal and external fluid pressure on internal & external surfaces. But now, there is a centrifugal force acting everywhere as a "body-force". A body-force is a gravity-like force that acts at every point in the body. It should be noted that the pressure force acts only on the surface and so it is used as a boundary condition when we solve for displacements, strains, and stresses. A body-force, on the other hand, enters the differential equation directly. Let us see how...





ω = angular velocity

ρ = mass density = kg/m^3

$\rho\omega^2 r$ = centrifugal force per unit volume

Balance forces in the radial direction:

$$(\sigma_r + d\sigma_r)(r+dr)\,l\,d\theta + \rho\omega^2 r\,l\,dr\,d\theta = \sigma_r\, r\,l\,d\theta + 2\,\sigma_\theta\,l\,dr\,\underbrace{\sin\frac{d\theta}{2}}_{\simeq \frac{d\theta}{2}}$$

$$\Rightarrow \sigma_r\,\cancel{r\,l\,d\theta} + l\,\sigma_r\,dr\,d\theta + r\,l\,d\sigma_r\,d\theta + \overbrace{\cancel{l\,d\sigma_r\,dr\,d\theta}}^{\to 0\ \text{(III order term)}} + \rho\omega^2 r\,l\,dr\,d\theta$$

$$= \sigma_r\,\cancel{r\,l\,d\theta} + \sigma_\theta\,l\,dr\,d\theta$$

$$\Rightarrow \left(\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r\right)(l\,dr\,d\theta) = 0$$

$$\Rightarrow \quad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0$$

Recall that $\varepsilon_r = \dfrac{du}{dr}$ $\left\{ \text{from } \dfrac{\text{change in length}}{\text{original length}} \text{ concept.} \right.$

$\varepsilon_\theta = \dfrac{u}{r}$

and $\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \dfrac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix}$ $\left\{ \text{from Hooke's law} \right.$

By substituting

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right) = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\nu\varepsilon_r + \varepsilon_\theta\right) = \frac{E}{1-\nu^2}\left(\nu\frac{du}{dr} + \frac{u}{r}\right)$$

$$\frac{d\sigma_r}{dr} = \frac{E}{1-\nu^2}\left(\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \frac{\nu u}{r^2}\right)$$

into $\dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0$, we get:

$$\boxed{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0}$$ Governing differential equation

This has the following soln.

$$u = c_1 r + \frac{c_2}{r} - \frac{1-\nu^2}{E}\frac{\rho\omega^2 r^3}{8}.$$ $\left\{ \text{verify by substitution if you wish.} \right.$

c_1 and c_2 are constants to be evaluated from the boundary conditions:

1) $\sigma_r = 0$ @ $r = r_i$
2) $\sigma_r = 0$ @ $r = r_o$.

Let us now use the boundary conditions to get c_1 and c_2.

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right).$$

$$= \frac{E}{1-\nu^2}\left(c_1 - \frac{c_2}{r^2} - \frac{1-\nu^2}{E}\frac{3}{8}\rho\omega^2 r^2 + c_1\nu + \frac{c_2\nu}{r^2} - \frac{1-\nu^2}{8E}\rho\nu\omega^2 r^2\right)$$

$$= \frac{E}{1-\nu^2}\left((1+\nu)c_1 + \frac{c_2}{r^2}(\nu-1) - \frac{(1-\nu^2)\rho\omega^2 r^2}{8E}(3+\nu)\right)$$

$\sigma_r = 0$ for $r = r_i$ and for $r = r_o$.

$$\Rightarrow \quad (1+\nu)c_1 + \frac{c_2}{r_i^2}(\nu-1) = \frac{(1-\nu^2)\rho\omega^2 r_i^2(3+\nu)}{8E}$$

and
$$(1+\nu)c_1 + \frac{c_2}{r_o^2}(\nu-1) = \frac{(1-\nu^2)\rho\omega^2 r_o^2(3+\nu)}{8E}$$

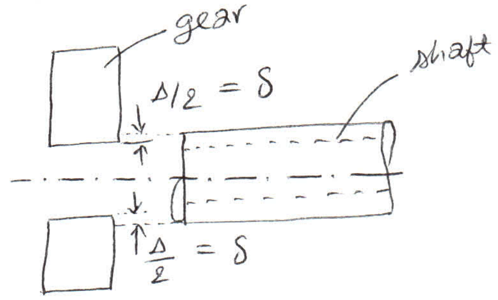
Solve for c_1 and c_2 and substitute into σ_r and σ_θ equations to get:

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(r_o^2 + r_i^2 - \frac{r_o^2 r_i^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(r_o^2 + r_i^2 + \frac{r_o^2 r_i^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right).$$

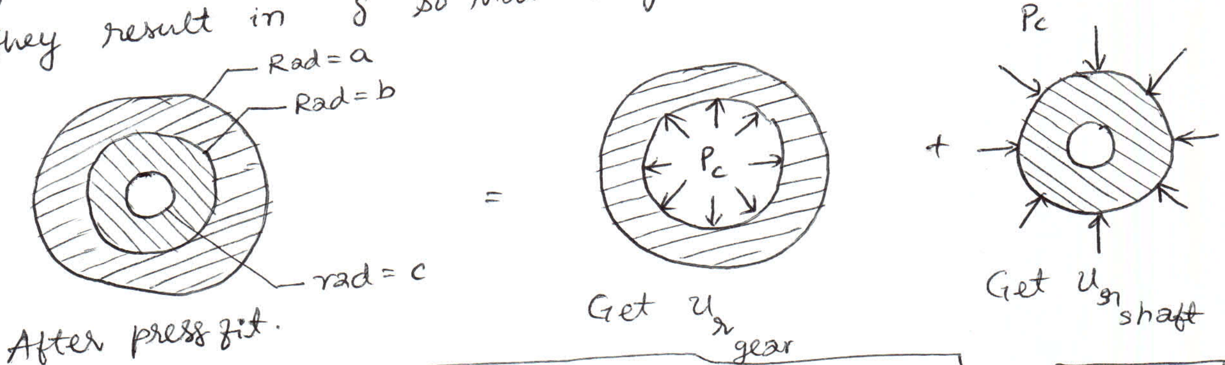
To really understand this solution, we should consider one more application of this, namely, "interference fits".

Let us say that we want to mount a gear on a hollow shaft by pressing the gear onto the shaft with force. It is called an "interference fit" or a "press fit". For this to work, the inner radius of the gear must be a little smaller than the outer radius of the shaft. See below.



$$\delta = \text{radial interference} = r_{\substack{\text{shaft}\\(\text{out})}} - r_{\substack{\text{gear}\\(\text{in})}}$$

When we do this press-fit, there will be an "interface" pressure between the gear and the shaft. Let us call it P_c. P_c is like an internal pressure on the gear and makes it expand some. P_c is like an external pressure on the shaft and makes it contract some. Together, they result in δ so that they fit together nicely.



$$\boxed{\delta = \left(u_{r_{gear}} @ r = b\right) - \left(u_{r_{shaft}} @ r = b\right)}$$

condition to find P_c for given δ