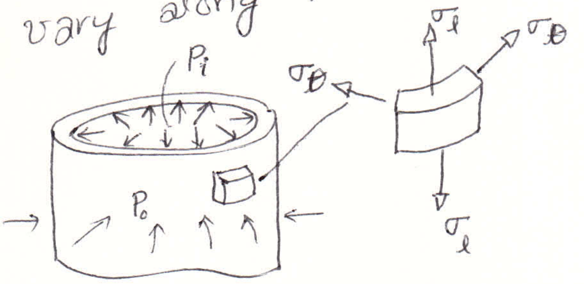
Thin and thick cylindrical shells

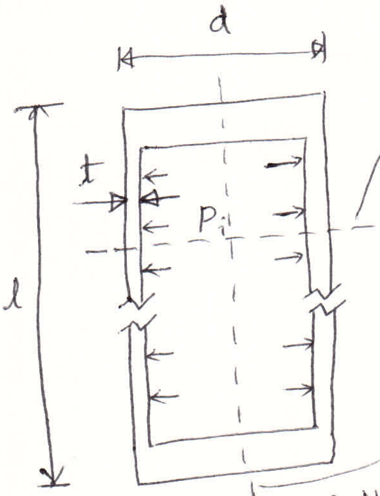
cylindrical shells have many applications in mechanical design. Pressure vessels, flywheels, and interference fits on shafts are some examples. Let us begin with pressure vessels. From the point of view of our discussion of strength of materials, shells are good "intermediate" elements between bars, beams, and columns and general 2-D and 3-D solids.

Thin cylindrical shells (thin-walled pressure vessels)

These are easy to analyze with the help of FBDs. Since we assume that the shell is very thin compared to its diameter d, and length l, the stresses in it do not vary along the radius. Due to symmetry, σ_θ, the tangential (or circumferential) stress is the same along the circular periphery. Longitudinal stress, σ_l, exists if the ends of the cylinder are closed. Radial stress, σ_r, is zero for thin shells and hence is not shown in the figure.

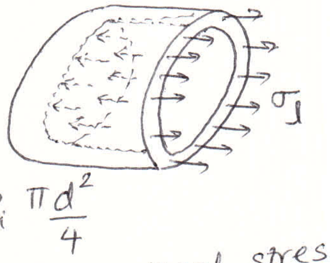


Loads internal pressure = P_i
external pressure = P_o



Side view of the cylinder

a cut here shows

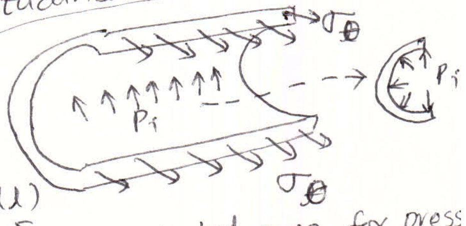


$$\Sigma F_l = 0$$

$$\Rightarrow (\sigma_l)(\pi d t) = P_i \frac{\pi d^2}{4}$$

$$\Rightarrow \sigma_l = \frac{P_i d}{4t} = \text{longitudinal stress}$$

a cut here shows



Now,

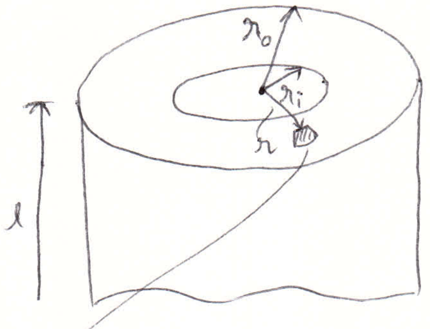
$$(\sigma_\theta)(2lt) = P_i(dl)$$

projected area for press

$$\Rightarrow \sigma_\theta = \frac{P_i d}{2t} = \text{circumferential stress}$$

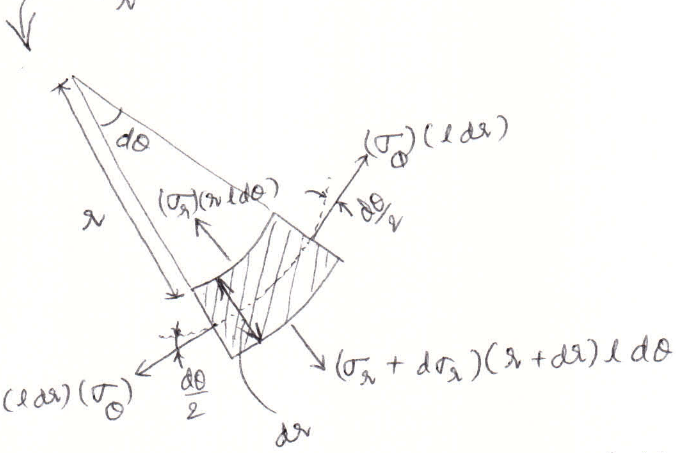
Thick cylindrical shells

In these, radial stress σ_r is not assumed to be zero. Both σ_r and σ_θ are dependent on r. That is, they vary along the radial direction. σ_l, on the other hand, can be assumed to be not dependent on r because length may be much bigger compared to inner and outer radii.



Let us use $u(r)$ to denote radial displacement at a point which is at a distance r from the center. Note that when there is an internal pressure, the cylinder expands.

$$\varepsilon_r = \text{radial strain} = \frac{\text{change in length}}{\text{original length}} = \frac{du}{dr}$$



$$\varepsilon_\theta = \text{circumferential strain}$$
$$= \frac{\text{change in length}}{\text{original length}}$$
$$= \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

Force balance on the differential element shown above gives the governing differential equation.

$(\sigma_\theta)(l\,dr)\cos\frac{d\theta}{2}$ = itself

$\sum$ circumferential forces $= 0 \Rightarrow$

$\sum$ radial forces $= 0 \Rightarrow (\sigma_r + d\sigma_r)(r + dr)\,l\,d\theta = \sigma_r\, r\,l\,d\theta + 2\sigma_\theta\, l\,dr\, \overset{\approx \frac{d\theta}{2}}{\sin\frac{d\theta}{2}}$

$\Rightarrow \sigma_r \cancel{r\,l\,d\theta} + l\sigma_r\, dr\, d\theta + d\sigma_r\, r\,l\,d\theta + \underbrace{d\sigma_r\, \cancel{dr}\, l\, d\theta}_{\substack{0 \\ \text{(third order term)}}} = \sigma_r \cancel{r\,l\,d\theta} + \sigma_\theta\, l\,dr\,d\theta$

Divide by $l\,dr\,d\theta$ to get

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad \leftarrow \text{governing eqn. in stresses.}$$

Let us now convert the differential equation from stresses to radial displacement, u. To do that, we express stresses in terms of strain, and then strains in terms of displacement.

We already have, $\varepsilon_r = \dfrac{du}{dr}$ and $\varepsilon_\theta = \dfrac{u}{r}$.

To get stresses in terms of strains, note Hooke's law.

$$\left.\begin{aligned} \varepsilon_r &= \frac{\sigma_r}{E} - \frac{\nu\sigma_\theta}{E} \\ \varepsilon_\theta &= -\nu\frac{\sigma_r}{E} + \frac{\sigma_\theta}{E} \end{aligned}\right\} \text{the second term is due to Poisson's effect.}$$

$$\Rightarrow \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = E\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix}$$

$$\Rightarrow \sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right) = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right) = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substitution of these into the governing equation of the previous page gives:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad \text{becomes}$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \quad \longleftarrow \text{Governing differential equn. in } u.$$

Solution $\quad u = c_1 r + \dfrac{c_2}{r}$

* verify by substituting back. c_1 and c_2 are determined from the boundary conditions.

Boundary conditions come from internal and external pressures because they are equivalent to radial stresses there. That is,

$$\left.\begin{aligned}\sigma_r &= -p_i \quad @ \quad r = r_i\\ \sigma_r &= -p_o \quad @ \quad r = r_o\end{aligned}\right\} \quad \text{-ve signs arise because positive pressure is a compressive stress.}$$

with $u = c_1 r + \frac{c_2}{r}$

$$\frac{du}{dr} = c_1 - \frac{c_2}{r^2}$$

gives stresses $\sigma_r = \underbrace{\frac{E}{1-\nu^2}\left(c_1 - \frac{c_2}{r^2} + \nu\left(c_1 + \frac{c_2}{r^2}\right)\right)}_{\frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)}$

$$\therefore \quad \sigma_r = \frac{E}{1-\nu^2}\left\{(1+\nu)c_1 + \frac{c_2}{r^2}(\nu - 1)\right\}$$

At $r = r_i$, $\quad (1+\nu)c_1 + \frac{c_2}{r_i^2}(\nu - 1) = -\frac{p_i(1-\nu^2)}{E} \quad$ — ①

At $r = r_o$, $\quad (1+\nu)c_1 + \frac{c_2}{r_o^2}(\nu - 1) = -\frac{p_o(1-\nu^2)}{E} \quad$ — ②

Solving for c_1 and c_2 using ① and ② and substitution into σ_r and σ_θ equns. gives

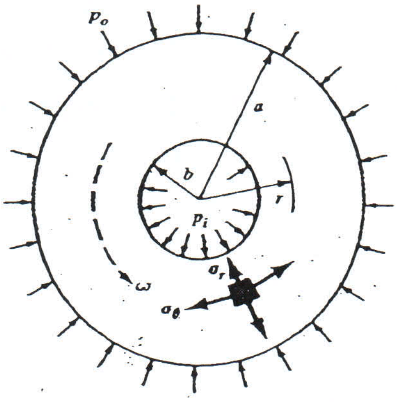
$$\left.\begin{aligned}\sigma_r &= \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right) - \frac{p_o r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)\\ \sigma_\theta &= \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right) - \frac{p_o r_o^2}{r_o^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right)\end{aligned}\right] \quad \text{Plot these in Matlab to see how they vary.}$$

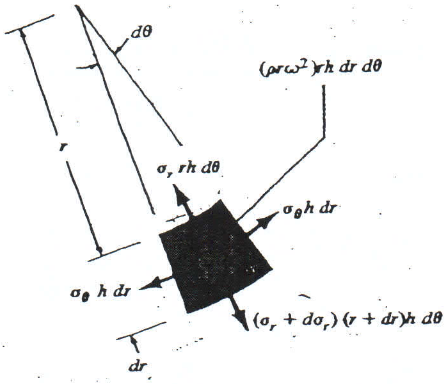
MEAM 310 Spring 2000

Thick-walled Pressure vessels and Spinning Disks

↖ flywheels)

& Shrink fits



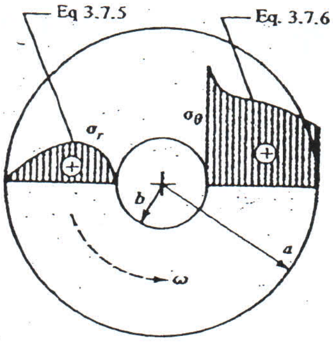


Balance of forces in radial direction and neglecting product of thr differential terms,

ρ = mass density

Governing equi. equn:

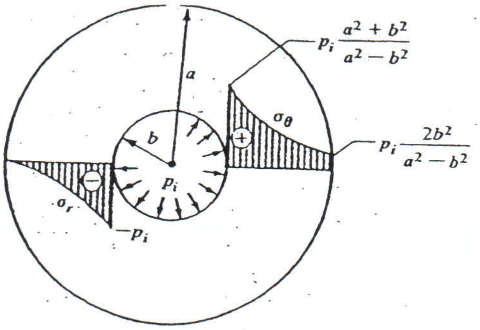
$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0$$



Spinning disk solution:

$$\sigma_r = \frac{3 + \nu}{8}\rho\omega^2\left(a^2 + b^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3 + \nu}{8}\rho\omega^2\left(a^2 + b^2 + \frac{a^2b^2}{r^2} - \frac{1 + 3\nu}{3 + \nu}r^2\right)$$



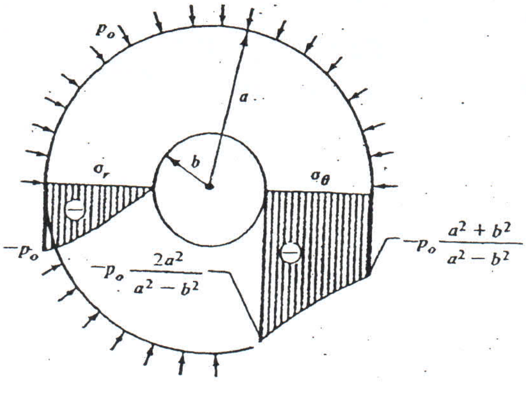
(a) Internal pressure

Pressure vessel solution:

|← internal pressure →|← external pressure →|

$$\sigma_r = \frac{p_ib^2}{a^2 - b^2}\left(1 - \frac{a^2}{r^2}\right) - \frac{p_oa^2}{a^2 - b^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{p_ib^2}{a^2 - b^2}\left(1 + \frac{a^2}{r^2}\right) - \frac{p_oa^2}{a^2 - b^2}\left(1 + \frac{b^2}{r^2}\right)$$



(b) External pressure

Linear superposition solution

Shrink fitting of two cylinders.
Interface radius c and interface pressure P_c are both unknowns.