ME237/NE211, IISc
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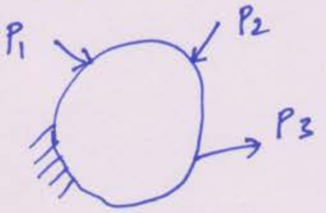
Castigliano's theorems

I. $\dfrac{\partial(SE)}{\partial d} = \underset{(\text{force})}{F}$ II. $\dfrac{\partial(SE)}{\partial F} = \underset{(\text{displacement})}{d}$

$\dfrac{\partial(SE)}{\partial \phi} = \underset{(\text{moment or torque})}{M}$ $\dfrac{\partial(SE)}{\partial M} = \underset{(\text{rotation or twist})}{\phi}$

Proof of Castigliano's II theorem:

By linear combination of loads P_1, P_2, and P_3 on an elastic body, we have



$$d_1 = c_{11} P_1 + c_{21} P_2 + c_{31} P_3$$
$$d_2 = c_{12} P_1 + c_{22} P_2 + c_{32} P_3$$
$$d_3 = c_{13} P_1 + c_{23} P_2 + c_{33} P_3$$

SE = strain energy = work done quasi-statically

$$= \frac{1}{2} P_1 d_1 + \frac{1}{2} P_2 d_2 + \frac{1}{2} P_3 d_3$$
$$= \frac{1}{2} P_1 (c_{11} P_1 + c_{21} P_2 + c_{31} P_3) + \frac{1}{2} P_2 (c_{12} P_1 + c_{22} P_2 + c_{32} P_3)$$
$$+ \frac{1}{2} P_3 (c_{13} P_1 + c_{23} P_2 + c_{33} P_3)$$

$$\frac{\partial(SE)}{\partial P_1} = \left(c_{11} P_1 + \frac{1}{2} c_{21} P_2 + \frac{1}{2} c_{31} P_3\right) + \left(\frac{1}{2} c_{12} P_2\right) + \left(\frac{1}{2} c_{13} P_3\right)$$

By using Maxwell's reciprocity theorem,

$$\frac{\partial(SE)}{\partial P_1} = c_{11} P_1 + c_{21} P_2 + c_{31} P_3 = d_1 \quad \text{(QED)}$$

General proof for n loads, $P_1, P_2, \ldots, P_n$

For n forces, $P_1, P_2, \ldots, P_n$, we have:

$$d_i = \sum_{j=1}^{n} c_{ji} P_j \quad \text{and} \quad SE = \frac{1}{2} \sum_{i=1}^{n} P_i d_i = \frac{1}{2} \sum_{i=1}^{n} P_i \left(\sum_{j=1}^{n} c_{ji} P_j\right)$$

$$\frac{\partial(SE)}{\partial P_k} = \frac{\partial\left(\frac{1}{2}\sum_{i=1}^{n} P_i \left(\sum_{j=1}^{n} c_{ji} P_j\right)\right)}{\partial P_k} = \frac{\partial\left(\frac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{n} P_i \left(\sum_{i=1}^{n} c_{ji} P_j\right)\right)}{\partial P_k} + \frac{\partial\left(\frac{1}{2} P_k \sum_{j=1}^{n} c_{jk} P_j\right)}{\partial P_k}$$

$$\Rightarrow \frac{\partial(SE)}{\partial P_k} = \left(\frac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{n} \underline{\underline{c_{ki}}} P_i\right) + \left(\frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n} \underline{\underline{c_{jk}}} P_j + c_{kk} P_k\right) = d_k \quad \text{(By Maxwell reciprocity)}. \quad \boxed{\text{QED}}$$