Coupled Electromechanics

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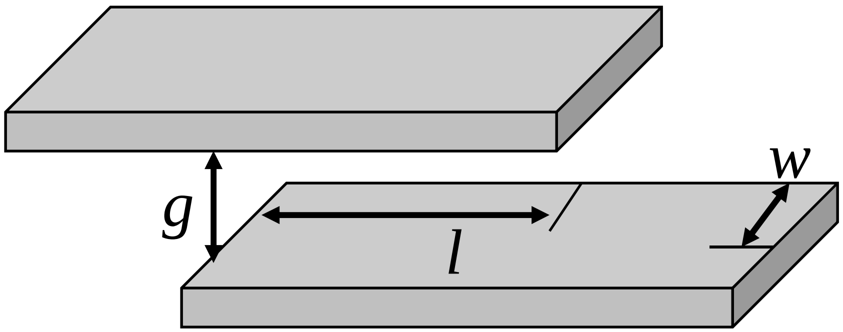
Computing the electrostatic force in a parallel-plate capacitor



ε_0 = permittivity of free space

V = applied voltage

C = capacitance

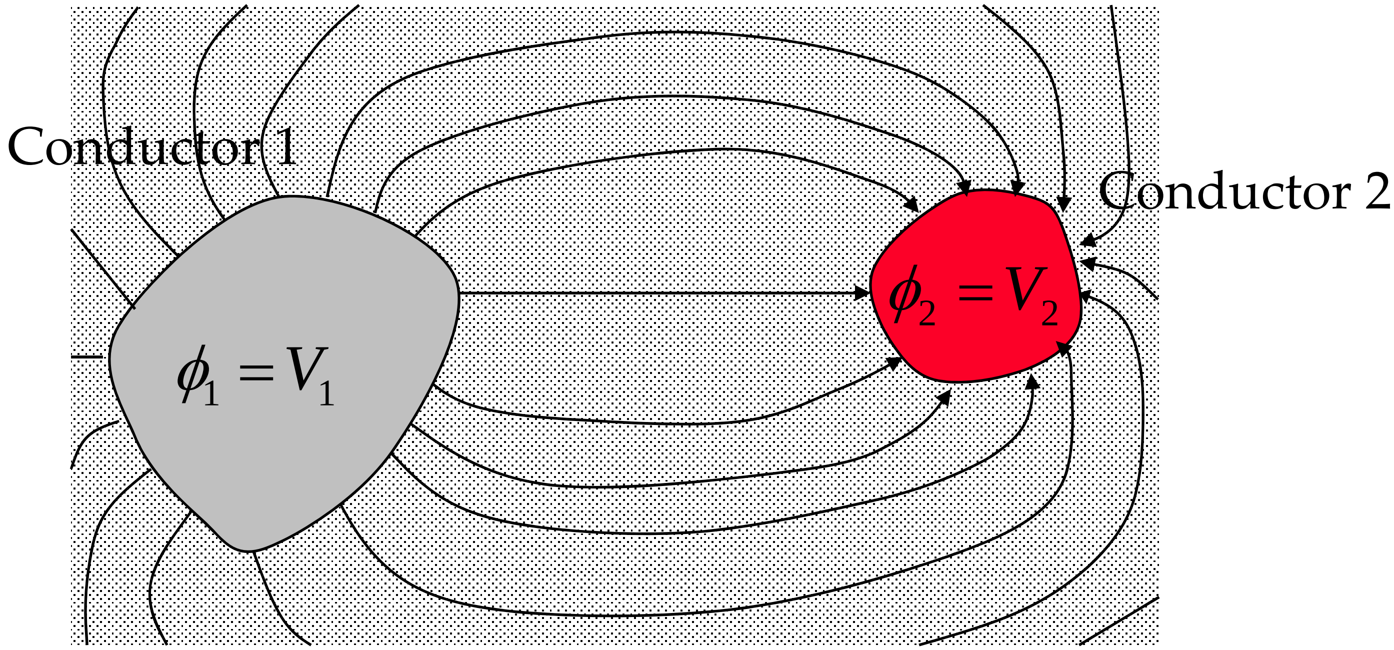
$$ESE_c = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_0 (wl)}{g}V^2$$ ← Electrostatic co-energy

$$F_l = -\frac{\partial ESE_c}{\partial l} = -\frac{1}{2}\frac{\varepsilon_0 w}{g}V^2$$ ← Force in the length direction

$$F_w = -\frac{\partial ESE_c}{\partial w} = -\frac{1}{2}\frac{\varepsilon_0 l}{g}V^2$$ ← Force in the width direction

$$F_g = -\frac{\partial ESE_c}{\partial g} = \frac{1}{2}\frac{\varepsilon_0 wl}{g^2}V^2 = \frac{1}{2}\frac{\varepsilon_0 A}{g^2}V^2$$ ← Force in the gap direction

Computing the electrostatic force in general 3-D problems



Electric potential = ϕ

Electric field = $-\nabla\phi$

Charge density = charge per unit area

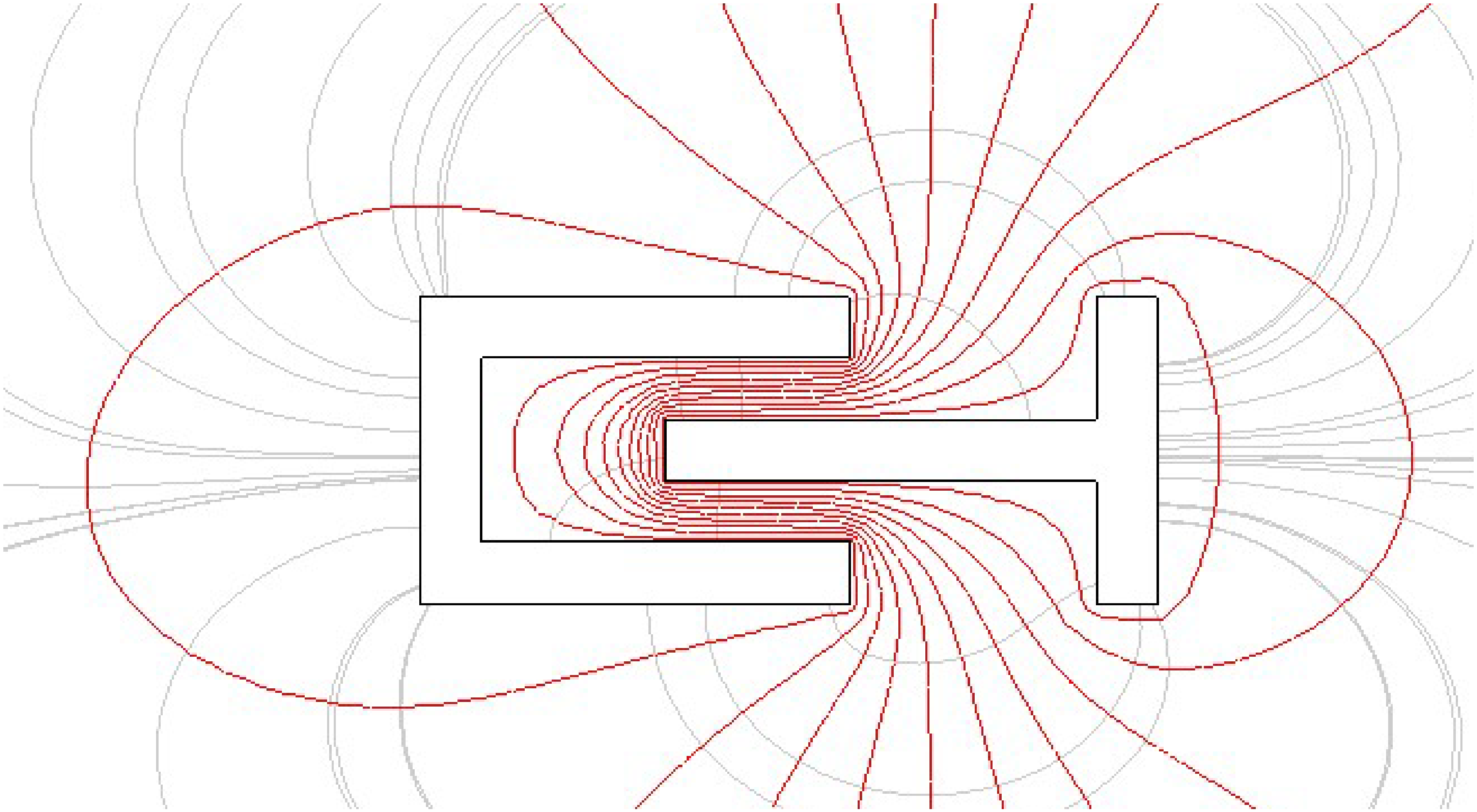
Surface normal

Electrostatic force = $\mathbf{F}_e = \dfrac{1}{2}\dfrac{\psi^2\hat{\mathbf{n}}}{\varepsilon}$

Permittivity of the intervening medium

It is a surface force (traction).

Electric field and iso-potential lines



Basics of electrostatics

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r}_{12}$$ Coulomb's law

$$\vec{E} = \lim_{q\to 0}\left(\frac{Qq}{4q\pi\varepsilon_0 r^2}\hat{r}_{12}\right) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}_{12}$$ Electric field due to a single charge

$$\phi = \frac{\text{work done}}{q} = -\int_{-\infty}^{P}\vec{E}\bullet d\vec{l}$$ Electrical potential (voltage)

$$\vec{E} = -\nabla\phi$$ Electric field is the negative of the gradient of the potential

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$$ Density of electric displacement vector

Gauss's law

$$\oint_S \vec{D}\bullet d\vec{S} = \int_V \psi_v dV = Q_v$$ Integral form

$$\vec{\nabla}\bullet\vec{D} = \psi_v \Rightarrow \nabla^2\phi = \psi_v$$ Differential form

Computing the electrostatic force

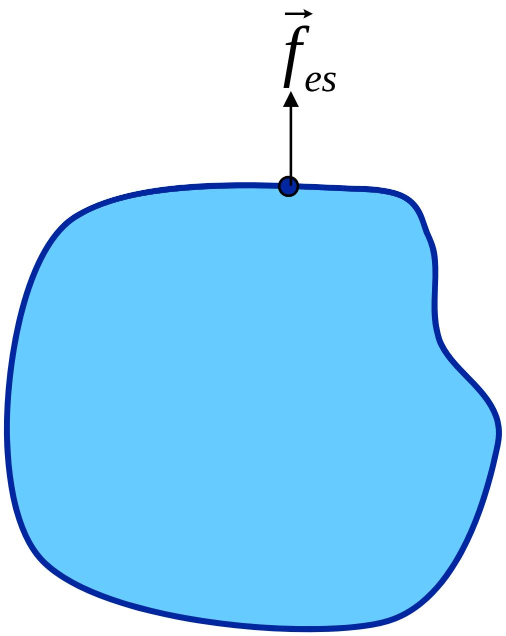
$$\psi_s = \hat{n} \bullet \vec{D} = \varepsilon\, \hat{n} \bullet \vec{E}$$ Surface charge density

$$E_n = \frac{\psi_s}{\varepsilon}$$ Normal component of the electric field

$$dF = E dq = \frac{\psi_s}{2\varepsilon} dq = \frac{\psi_s}{2\varepsilon} \psi_s dA$$

$$\Rightarrow \frac{dF}{dA} = \frac{\psi_s^2}{2\varepsilon}$$ Force along the surface normal

$$\vec{f}_{es} = \frac{\psi_s^2}{2\varepsilon} \hat{n}$$ Electrostatic force on the surface per unit area



Computing the electrostatic force (contd.)

Governing equations to solve for the charge density in the differential equation form:

$$\nabla^2 \phi = -4\pi\psi \qquad \text{On the conductors}$$

$$\nabla^2 \phi = 0 \qquad \text{In the intervening medium}$$

Plus, potentials on the conductors are specified.

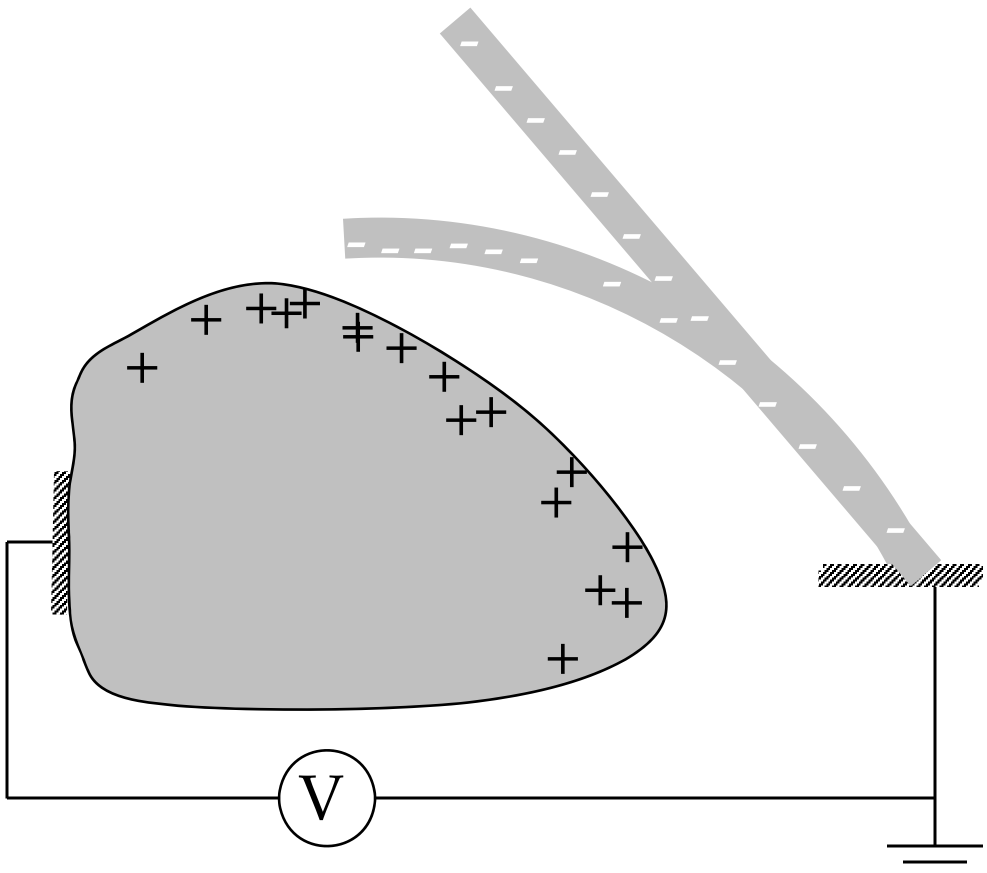
This is suited for FEM but sufficient intervening medium also needs to be meshed along with the interior of the conductors.

Governing equations to solve for the charge density in the integral equation form:

$$\phi(x) = \int_{Surfaces} \frac{\psi(x)}{\|x - x'\|} dS'$$

This is suited for BEM because only conductors' boundaries need to be meshed.

Static equilibrium of an elastic structure under electrostatic force



Electrostatic force = $\mathbf{F}_e = \dfrac{1}{2}\dfrac{\psi^2 \hat{\mathbf{n}}}{\varepsilon_0}$

Charge distribution causes electrostatic force of attraction between conductors

Electrostatic force deforms conductors

Deformation of conductors causes charges to re-distribute

Coupled governing equations of electro- and elasto- statics

$$\phi(x) = \int_{Surfaces} \frac{\psi(x)}{\left\| x - x' \right\|} dS' \quad \text{for } \delta\Omega s \text{ of all conductors}$$

$$\mathbf{f}_{te} = \frac{1}{2}\frac{\psi^2 \hat{\mathbf{n}}}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{\sigma} = 0 \quad \text{everywhere in } \Omega$$

$$\boldsymbol{\sigma}\hat{\mathbf{n}} = \mathbf{f}_{te} \quad \text{on } \delta\Omega$$

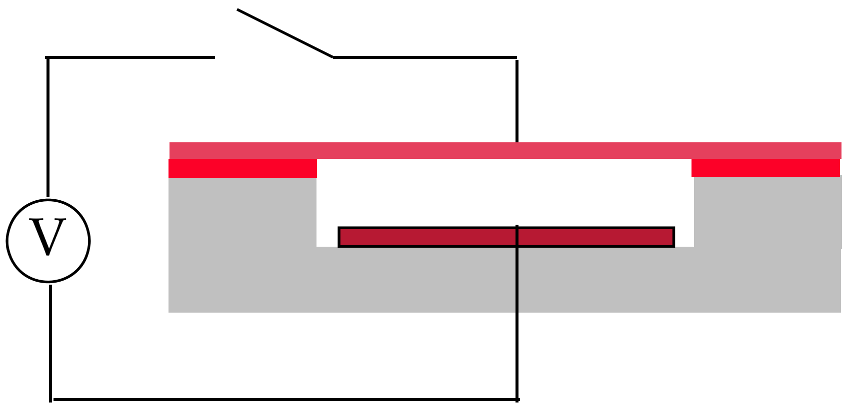
$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \delta\Omega_u$$

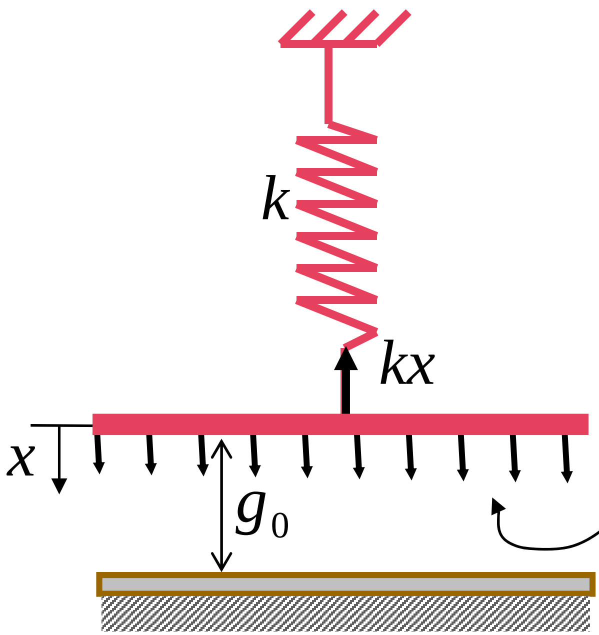
$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$$

A self-consistent solution is needed!

Start with 1-dof lumped model…





$$\frac{1}{2}\frac{\varepsilon_0 A}{(g_0 - x)^2}V^2$$

A = plate area

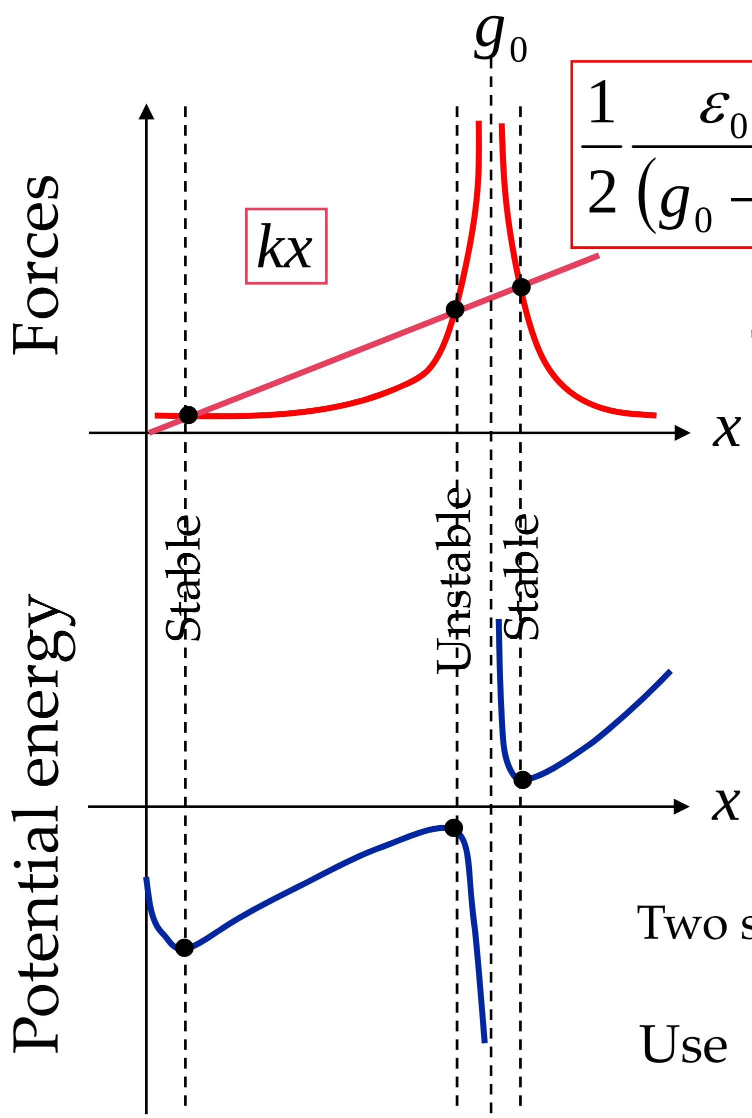
ε_0 = permittivity of free space

Static equilibrium

$$kx = \frac{1}{2}\frac{\varepsilon_0 A}{(g_0 - x)^2}V^2$$

A cubic equation!

Lumped 1-dof modeling of coupled electro- and elasto- static behavior



$$kx = \frac{1}{2}\frac{\varepsilon_0 A}{(g_0 - x)^2}V^2$$

Three solutions

$$\frac{\partial (PE)}{\partial x} = 0$$

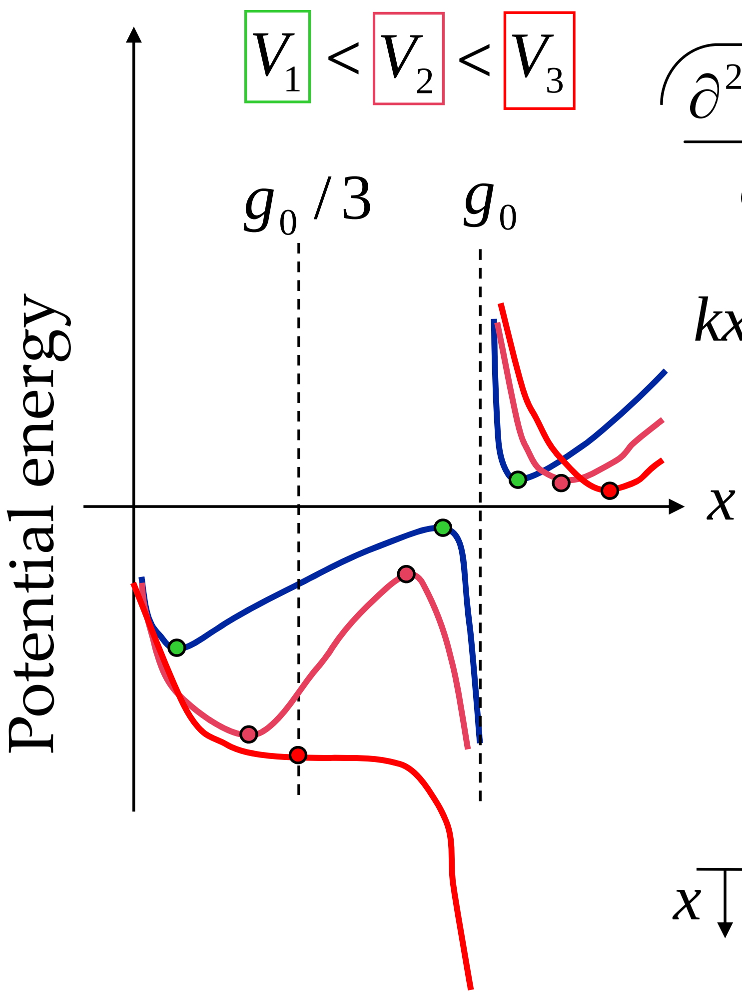
Potential energy

$$PE = \frac{1}{2}kx^2 - \frac{1}{2}\frac{\varepsilon_0 A}{(g_0 - x)}V^2$$

Two stable; one unstable; And, one infeasible

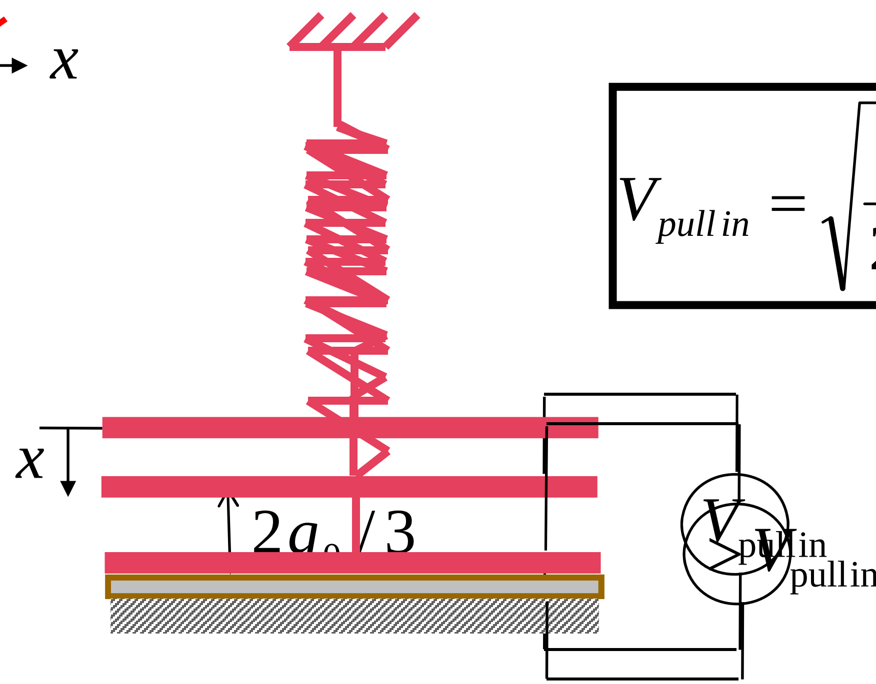
Use $\frac{\partial^2 (PE)}{\partial x^2} = k - \frac{\varepsilon_0 A V^2}{(g_0 - x)^3}$ to test stability.

Pull-in phenomenon



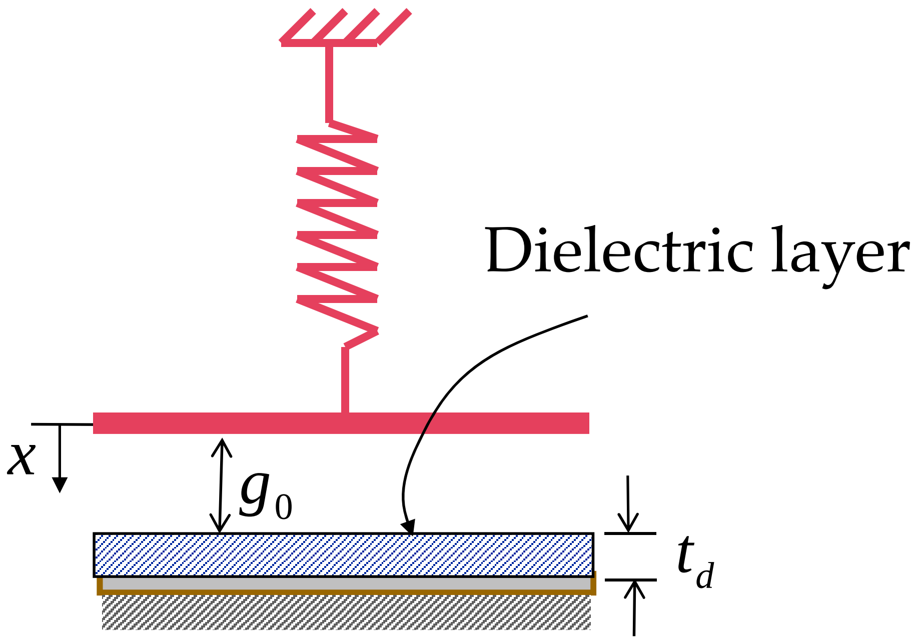
$$\overbrace{\frac{\partial^2 (PE)}{\partial x^2} = k - \frac{\varepsilon_0 A V^2}{(g_0 - x)^3}}^{\text{Condition for critical stability}} = 0 \Rightarrow V^2 = \frac{k(g_0 - x)^3}{\varepsilon_0 A}$$

$$kx = \frac{1}{2}\frac{\varepsilon_0 A}{(g_0 - x)^2} V^2 = \frac{k(g_0 - x)}{2} \Rightarrow x = \frac{g_0}{3}$$



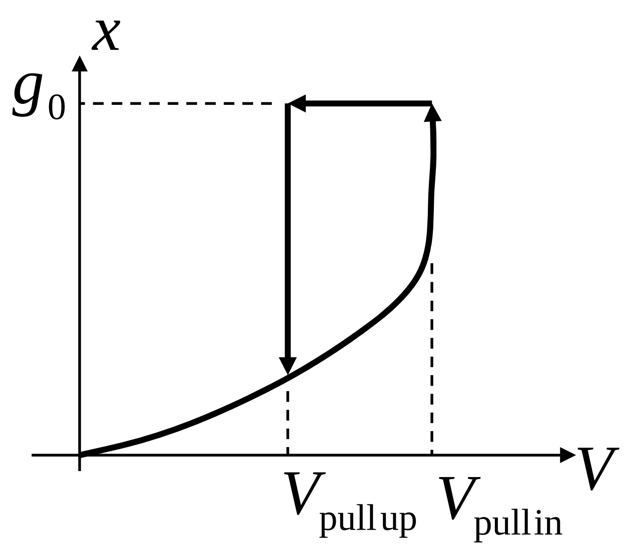
$$\boxed{V_{pull\,in} = \sqrt{\frac{8k\,g_0^3}{27\,\varepsilon_0 A}}}$$

With a dielectric layer: pull-up and hysteresis



$$V_{\text{pull in}} = \sqrt{\frac{8k}{27\varepsilon_0 A}\left(g_0 + \frac{t_d}{\varepsilon_r}\right)^3}$$

$$V_{\text{pull up}} = \sqrt{\frac{2k}{\varepsilon_0 A} g_0 \left(\frac{t_d}{\varepsilon_r}\right)^2}$$

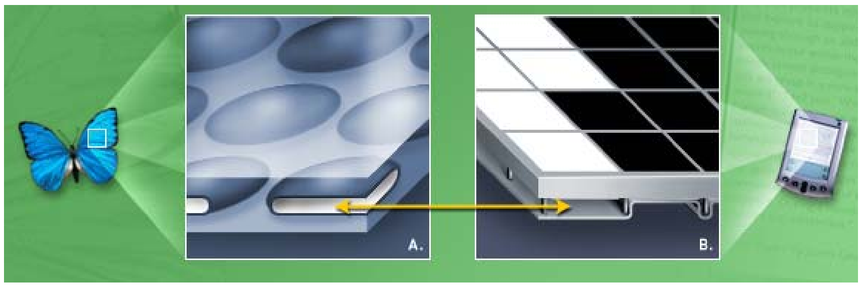


Pull-up voltage is found by equating the forces of spring and electrostatics at $x = g_0$.

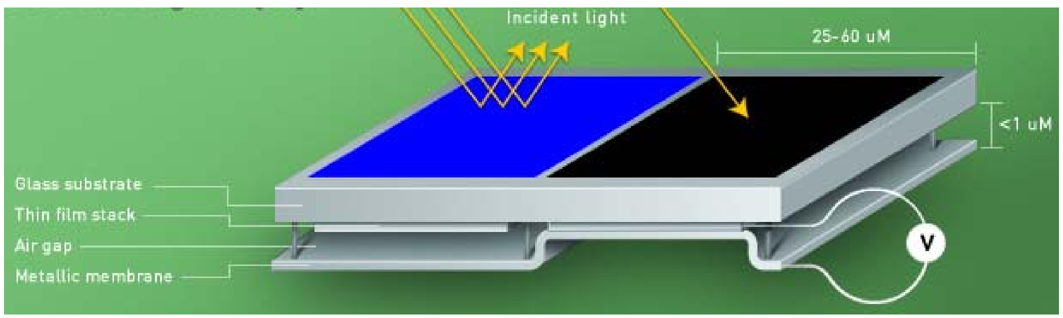
Gilbert, J. R., Ananthasuresh, G. K., and Senturia, S. D., “3-D Modeling and Simulation of Contact Problems and Hysteresis in Coupled Electromechanics,” presented at the IEEE-MEMS-96 Workshop, San Diego, CA, Feb. 11-15, 1996.

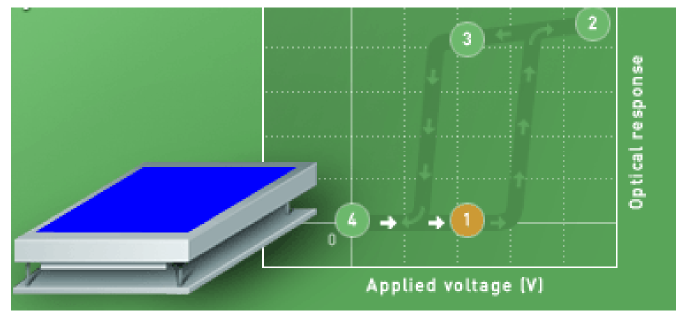
Pull-in phenomenon used in a display device

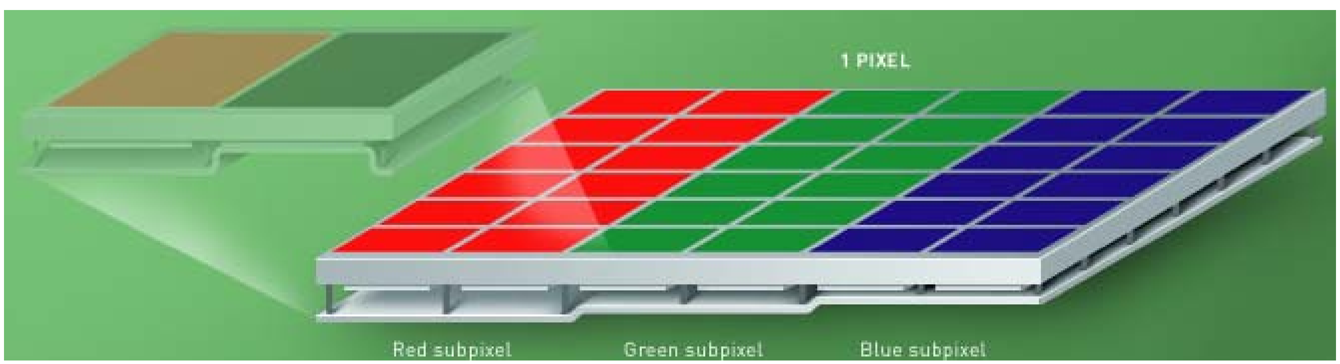
IRIDIGM™ www.iridigm.com (a QualComm acquisition)



Interference-modulation by electrostatic actuation of vertically moving membranes.

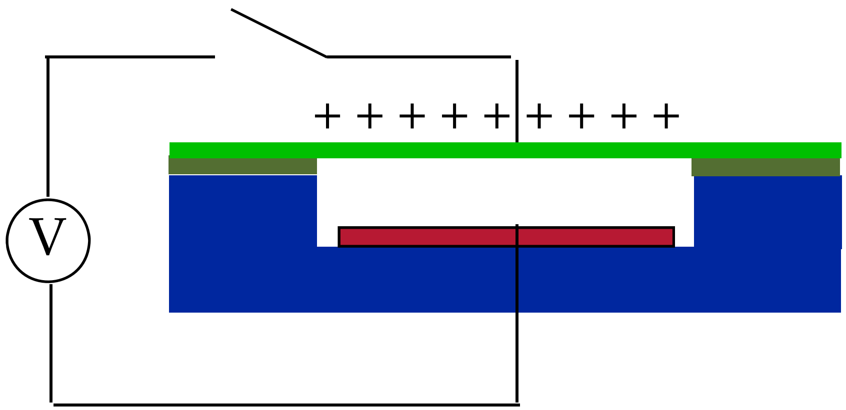






Iridigm became mirasol…

Distributed modeling of the electrostatically actuated beam

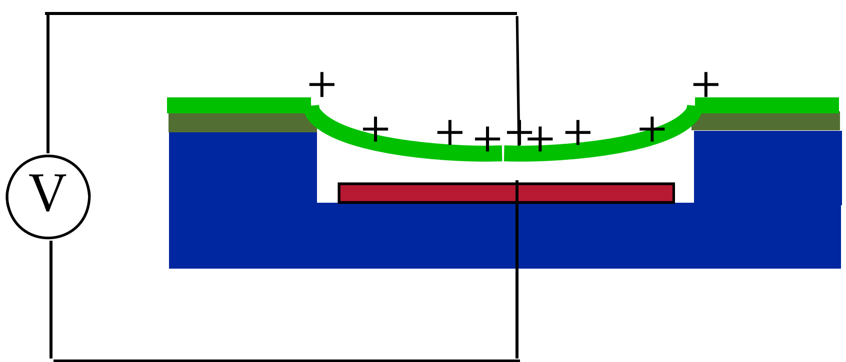


Finite element method

Finite difference method

FEM or FDM could be used to solve the nonlinear equation:

$$EI\frac{d^4u}{dx^4}-\frac{\varepsilon_0 wV^2}{2(g_0-u)^2}=0$$



Include the effects of residual stress as well:

$$EI\frac{d^4u}{dx^4}-\sigma_0 wt\frac{d^2u}{dx^2}-\frac{\varepsilon_0 wV^2}{2(g_0-u)^2}=0$$

A correction due to fringing field (edge and corner effects) is also included.

Solving the general 3-D problem

Boundary element method for the integral equation of electrostatics

$$\phi(x) = \int_{Surfaces} \frac{\psi(x)}{\|x - x'\|} dS'$$

Discretize the boundary surfaces into n panels.

Charge on k^th panel

$$p_k = \sum_{i=1}^{n} \frac{q_i}{a_i} \int_{panel_i} \frac{da'}{\|x' - x_k\|}$$

Area of k^th panel

Potential on k^th panel

Assemble to get:

$$\{\mathbf{p}\} = [\mathbf{P}]\{\mathbf{q}\} \quad \Rightarrow [\mathbf{C}]\{\mathbf{p}\} = \{\mathbf{q}\}$$

Finite element method for the differential equation of elastostatics

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}_e\}$$

$$\mathbf{f}_{\text{on panel k}} = \frac{(q_k / a_k)^2}{2\varepsilon} \hat{\mathbf{n}}_k$$

Solution approaches

Relaxation

-- iterate between the elastic and electrostatic domains.

-- converges except in the vicinity of pull-in voltage; but slow.

Surface Newton

-- compute sensitivities of surface nodes.

-- use a Newton step to update those nodes.

-- then, re-compute electrostatic force and internal deformations.

Direct Newton

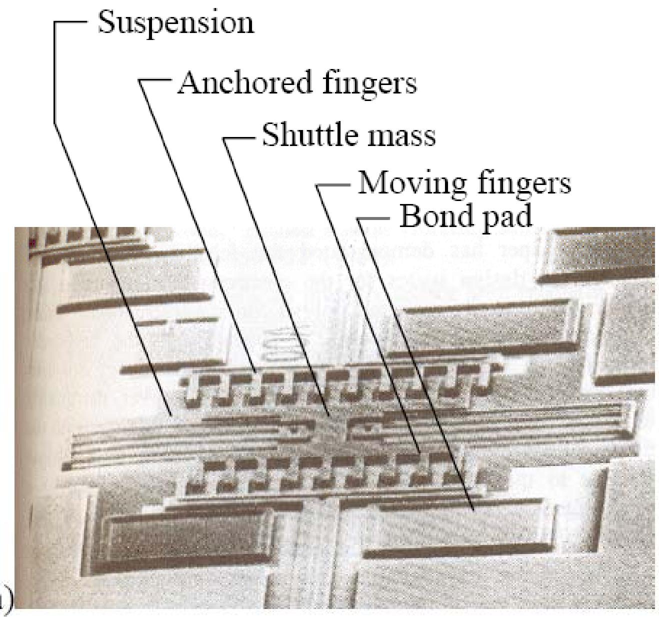
-- compute all derivatives to update charges and deformations.

$$\begin{bmatrix} \dfrac{\partial \mathbf{R}_M}{\partial \mathbf{U}} & \dfrac{\partial \mathbf{R}_M}{\partial \mathbf{q}} \\ \dfrac{\partial \mathbf{R}_E}{\partial \mathbf{U}} & \dfrac{\partial \mathbf{R}_E}{\partial \mathbf{q}} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{U} \\ \Delta \mathbf{q} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{R}_M \\ \mathbf{R}_E \end{Bmatrix}$$

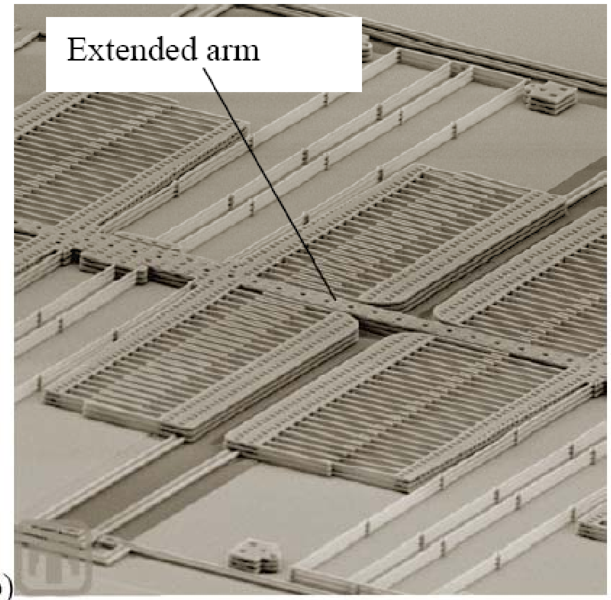
Residuals in mechanical and electrical domains

For example, see: G. Li and N. R. Aluru, "Linear, non-linear, and mixed-regime analysis of electrostatic MEMS," Sensors and Actuators, A 91, 2001, pp. 279-291, and references therein.

Comb-drive example



(a)

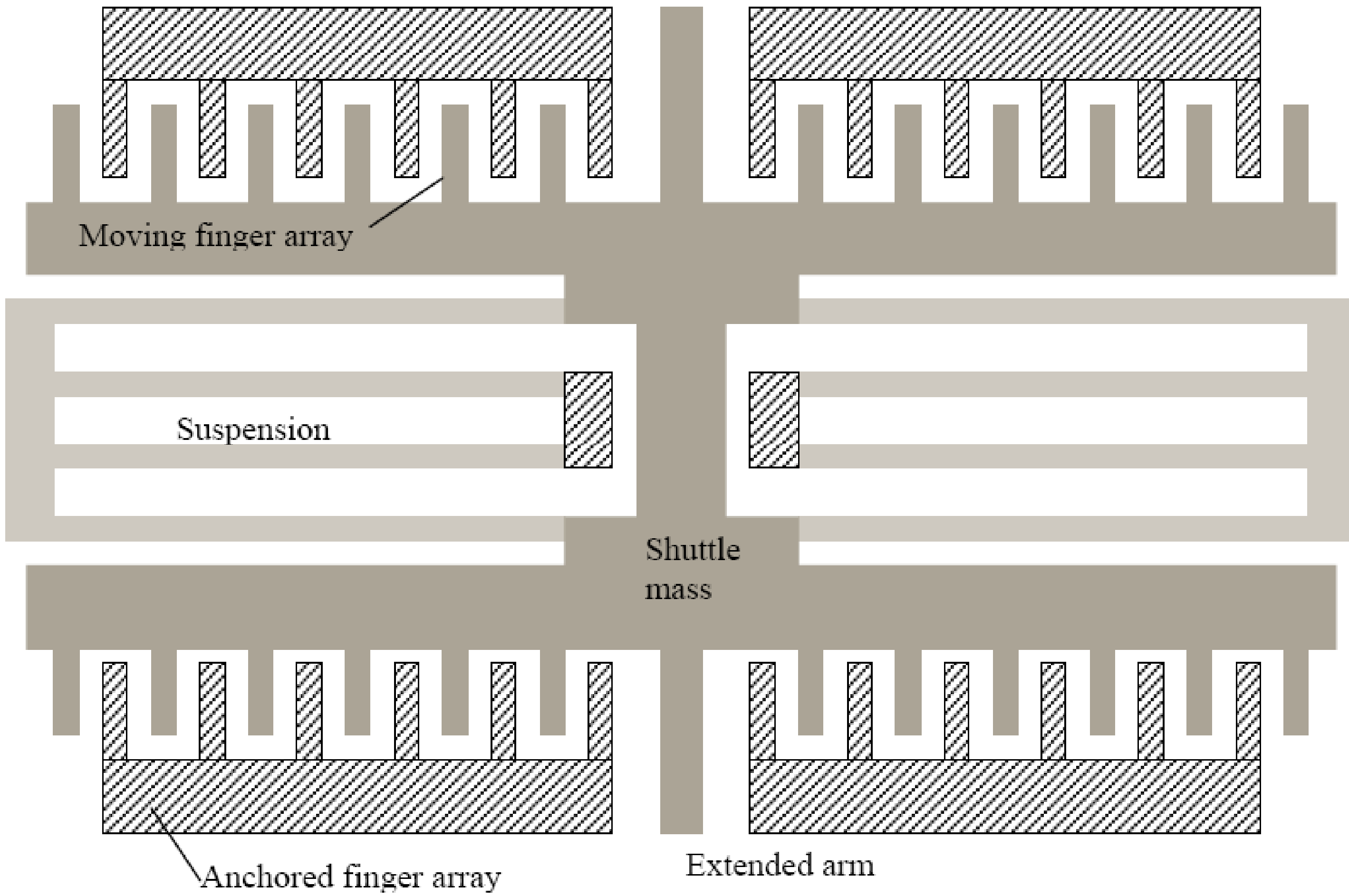


(b)

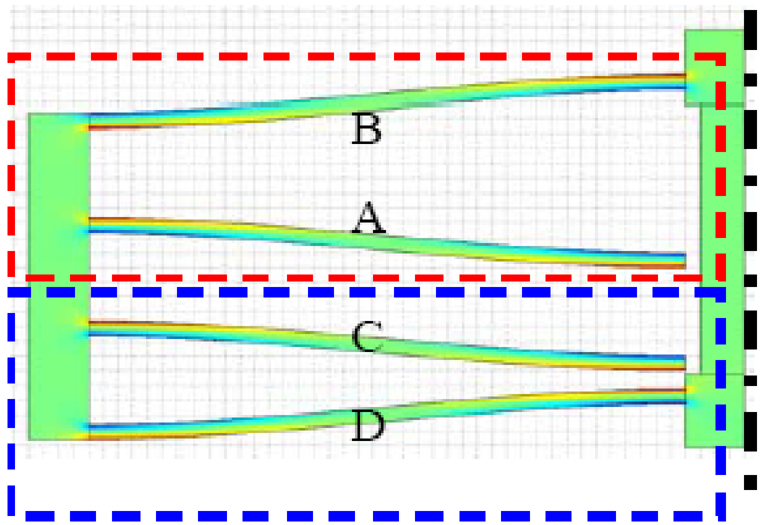
(a) Tang, W. C., Nguyen, C., Judy, M. W., and Howe, R. T., "Electrostatic Comb-Drive of Lateral Polysilicon Resonators," *Sensors and Actuators A*, 21 (1), 1990, pp. 328-331

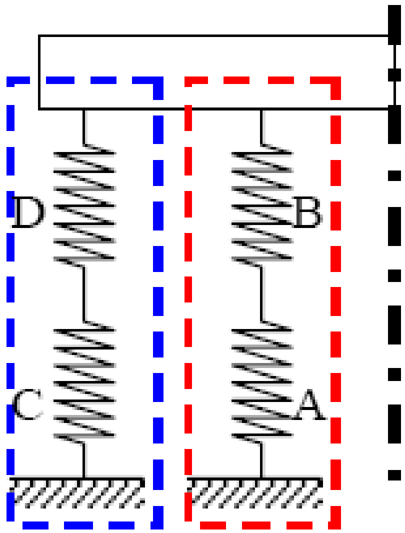
(b) http://mems.sandia.gov/scripts/images.asp (Sandia National Laboratories, New Mexico, USA)

Schematic of the comb-drive



Lumped mechanical stiffness





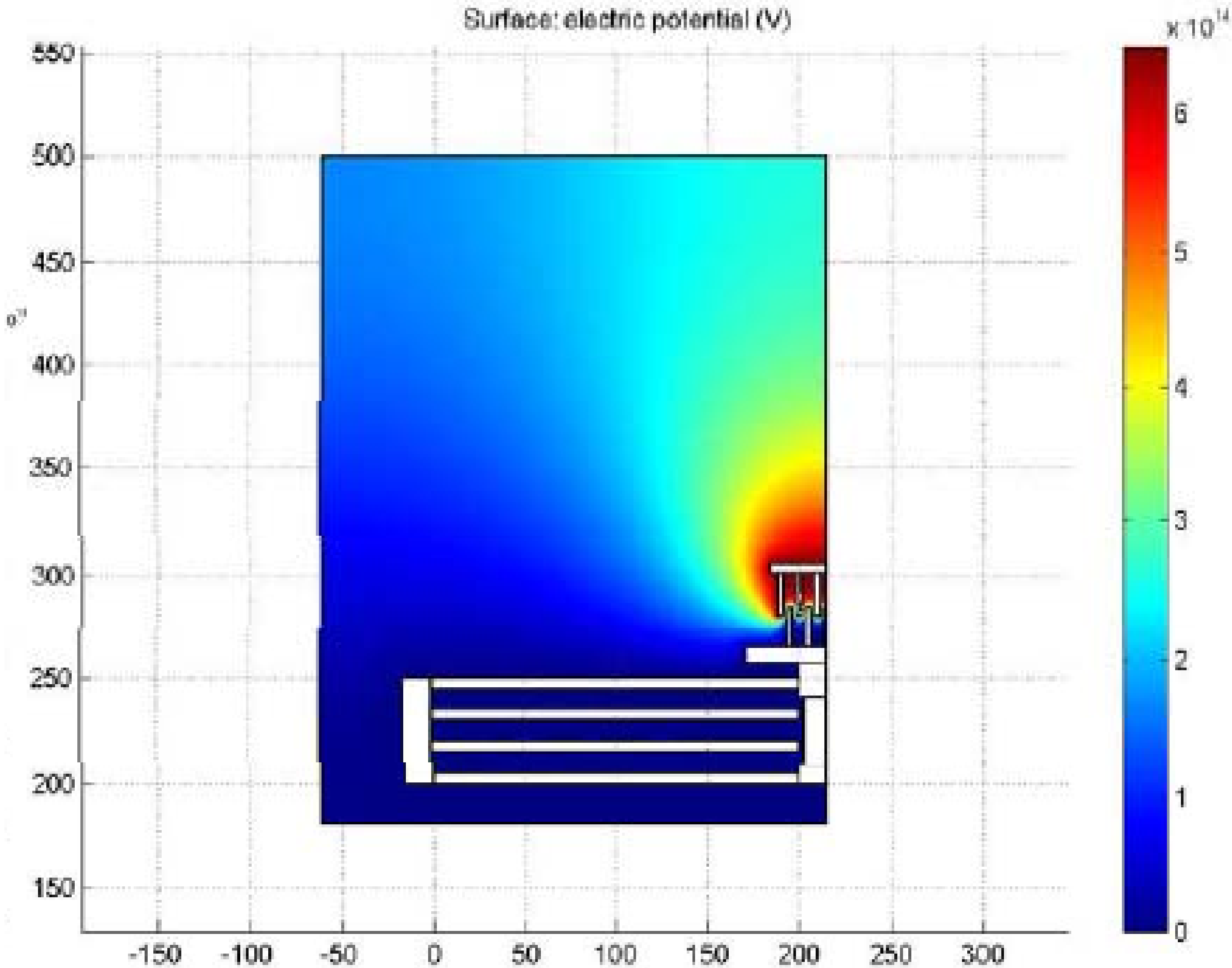
$$\delta = \frac{Fl^3}{12EI} = \frac{Fl^3}{12E\left(\frac{bh^3}{12}\right)} = \frac{Fl^3}{Ebh^3}$$

$$k = \frac{F}{\delta} = \frac{Ebh^3}{l^3}$$

$$k = \frac{Ebh^3}{l^3} = \frac{Etw^3}{l^3}$$

$$k_{\text{total}} = \frac{2Etw^3}{l^3}$$

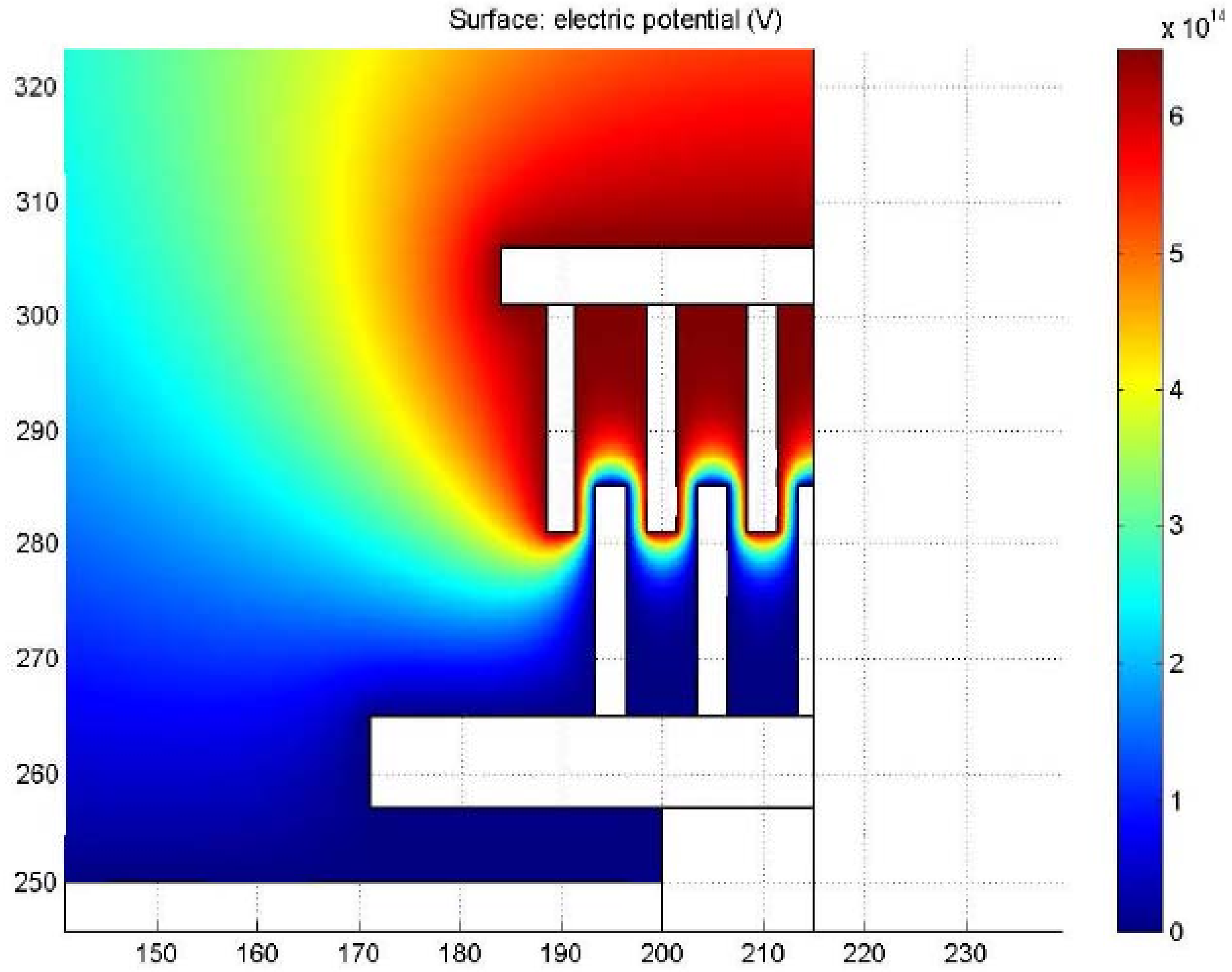
Electric potential solution



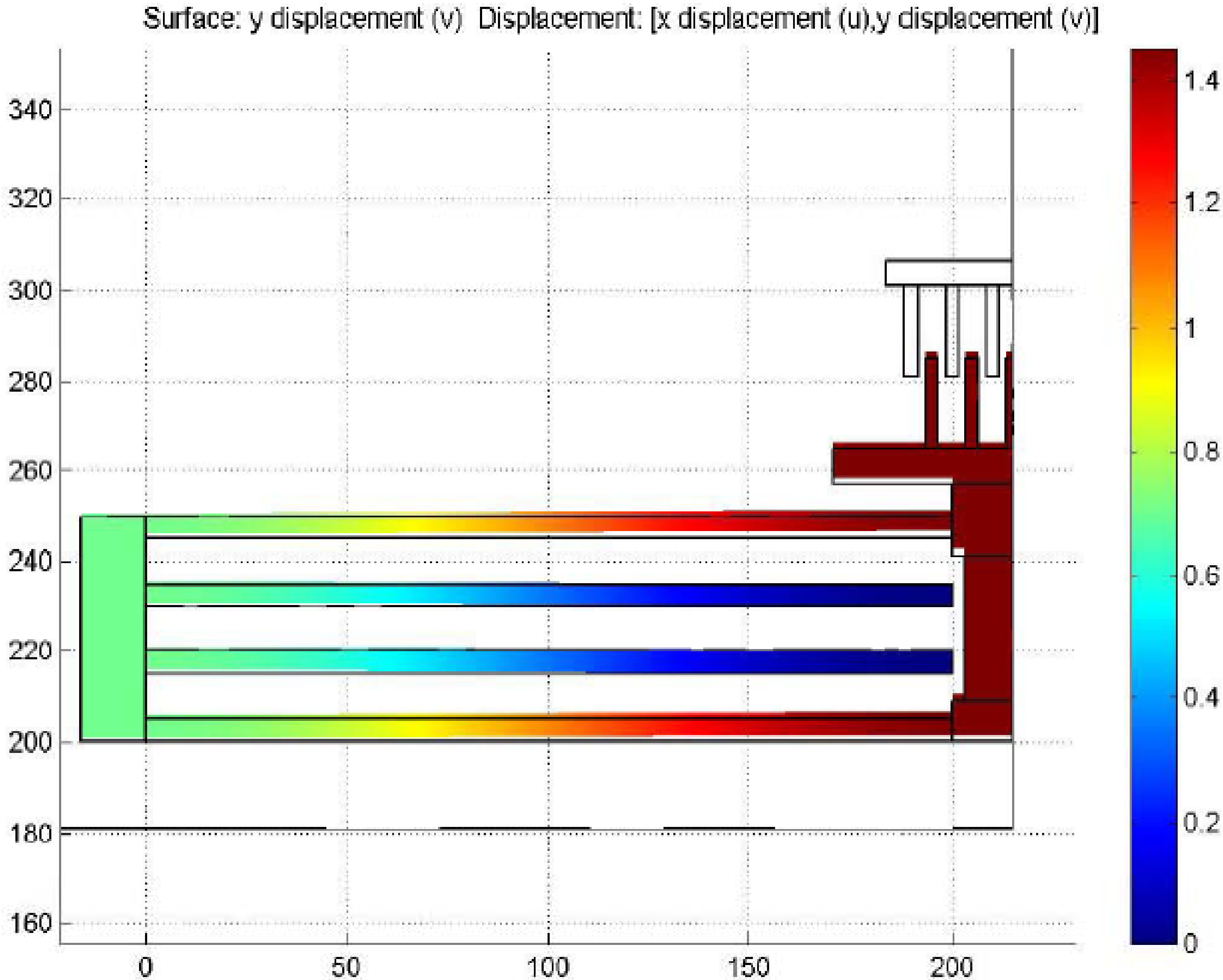
FEMLAB (old COMSOL) result

Close-up of the electric potential

FEMLAB (old COMSOL) result

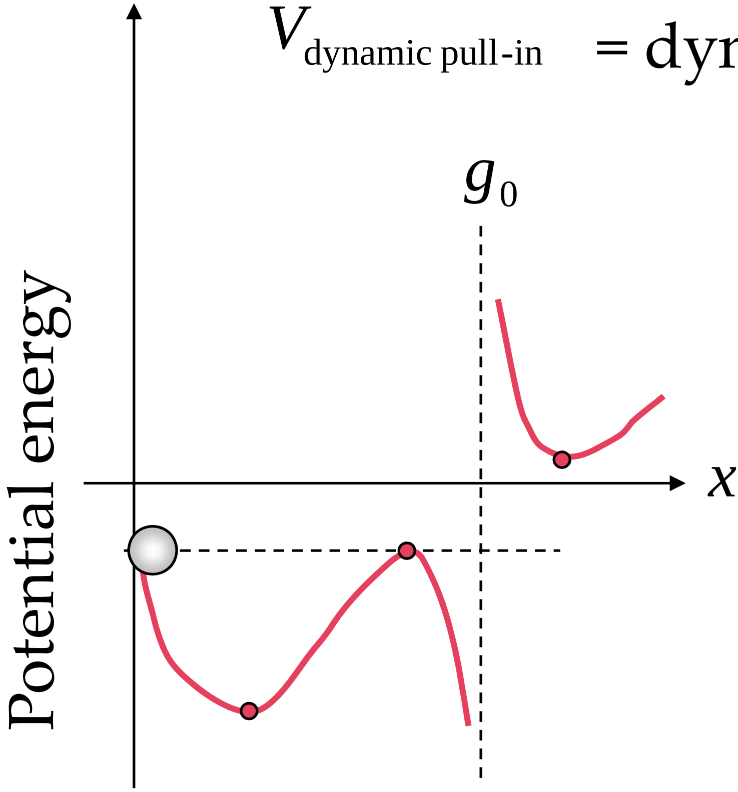


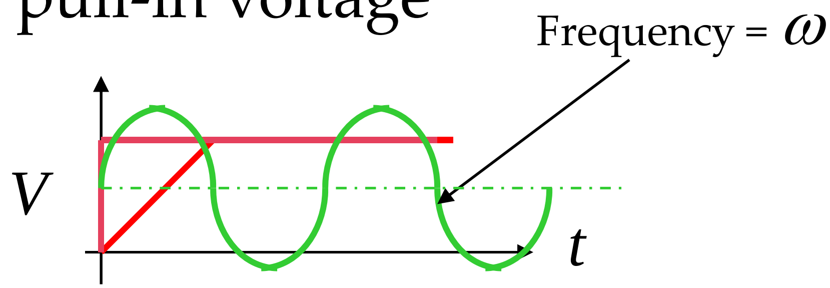
Deformation of the comb-drive



What about dynamic behavior?

$V_{\text{dynamic pull-in}}$ = dynamic pull-in voltage





Lumped 1-dof model

$$m\ddot{x} + kx = \frac{\varepsilon_0 A V^2}{2(g_0 - x)^2}$$

Beam model

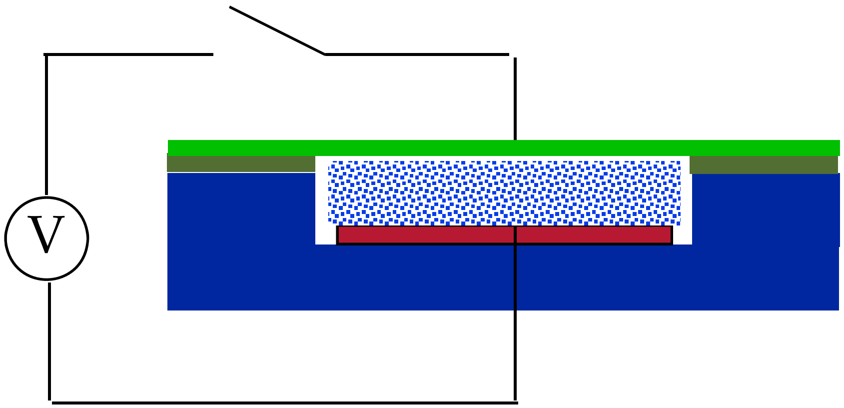
$$\rho\, w t\, \ddot{u} + EI\frac{d^4 u}{dx^4} - \frac{\varepsilon_0 w V^2}{2(g_0 - u)^2} = 0$$

$$\boxed{V^2 = (V_{dc} + V_{ac}\sin\omega t)^2 = V_{dc}^2 + 2V_{dc}V_{ac}\sin\omega t + V_{ac}^2\sin^2\omega t}$$

$$\underbrace{V_{ac}^2\sin^2\omega t}_{\text{Will contain a } 2\omega \text{ term!}}$$

So, the response will show two resonance at two frequencies.

Damping: squeezed film effects



Squeezed-film damping

Lumped 1-dof model

$$m\ddot{x} + b\dot{x} + kx = \frac{\varepsilon_0 A V^2}{2(g_0 - x)^2}$$

Beam model

$$\rho w t\, \ddot{u} + b\dot{u} + EI\frac{d^4 u}{dx^4} - \frac{\varepsilon_0 w V^2}{2(g_0 - u)^2} = 0$$

How do you obtain b ?

Use isothermal, compressible, narrow gap Reynolds equation to model the film of air beneath the beam/plate/membrane.
It is widely used in lubrication theory.
By analyzing this equation, we can extract the essence of damping as a lumped parameter – the so called "macromodeling".

Modeling squeezed film effects: isothermal Reynolds equation

Pressure distribution in the 2-D x-y plane

Gap varies in the x-y plane for a deformable structure (beam,plate, membrane)

$$\frac{\partial\left(p(x,y)\,g(x,y)\right)}{\partial t}=\frac{1}{12\nu}\nabla\cdot\left(p(x,y)\,g(x,y)^{3}\,\nabla p(x,y)\right)$$

Viscosity of air

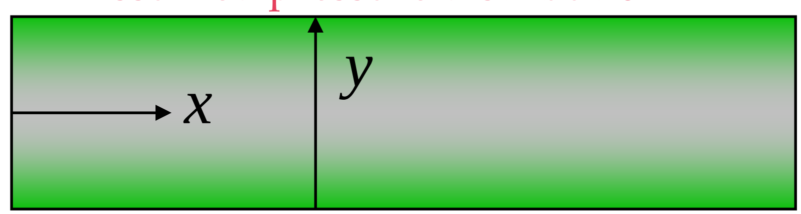
For lumped 1-dof modeling, we have a rigid plate. So, g does not depend on (x,y)

$$\Rightarrow\frac{\partial\left(p(x,y)\,g\right)}{\partial t}=\frac{g^{3}}{12\nu}\nabla\cdot\left(p(x,y)\,\nabla p(x,y)\right)=\frac{g^{3}}{12\nu}\left(\frac{1}{2}\nabla^{2}p^{2}(x,y)\right)$$

Assume further that pressure distribution is the same along the length of the plate so that it becomes a one dimensional problem.

$$\Rightarrow\frac{\partial\left(p(y)\,g\right)}{\partial t}=\frac{g^{3}}{12\nu}\left(\frac{1}{2}\nabla^{2}p^{2}(y)\right)$$

Assumed pressure distribution



S. D. Senturia, Microsystems Design, Kluwer, 2001.

Behavior with small displacements

Linearize $\dfrac{\partial\left(p(y)\,g\right)}{\partial t}=\dfrac{g^3}{12\nu}\left(\dfrac{1}{2}\nabla^2 p^2(y)\right)$ around (p_0, g_0):

$$p = p_0 + \delta p \qquad g = g_0 + \delta g$$

Also, use non-dimensional variables: $\xi = \dfrac{y}{w}$, $\hat{p} = \dfrac{\delta p}{p_0}$, $\hat{g} = \dfrac{\delta g}{g_0}$

width

$$\Rightarrow \frac{\partial \hat{p}}{\partial t} = \frac{g_0^2 p_0}{12\nu\, w^2}\frac{\partial^2 \hat{p}}{\partial \xi^2} - \frac{\partial \hat{g}}{\partial t} = \frac{g_0^2 p_0}{12\nu\, w^2}\frac{\partial^2 \hat{p}}{\partial \xi^2} - \frac{\dot{g}}{g_0}$$

Separation of spatial and temporal components: $\hat{p}(\xi, t) = \tilde{p}(\xi)\, e^{-\alpha t}$

$$\Rightarrow \frac{g_0^2 p_0}{12\nu\, w^2}\frac{\partial^2 \tilde{p}}{\partial \xi^2} + \alpha\, \tilde{p} = -\frac{\dot{g}}{g_0 e^{-\alpha t}}$$

Assume a sudden velocity impulse to the plate. Then, for t > 0, this term is zero.

(with displacement $x = x_0$)

Behavior with small displacements (contd.)

$$\boxed{\frac{g_0^2 p_0}{12\nu w^2}\frac{\partial^2 \tilde{p}}{\partial \xi^2} + \alpha\,\tilde{p} = 0 \Rightarrow \tilde{p} = A_n \sin\left(\sqrt{\sigma_n}\,\xi\right) + B_n \cos\left(\sqrt{\sigma_n}\,\xi\right)} \quad \boxed{\sigma_n = \frac{12\nu w^2 \alpha_n}{g_0^2 p_0}}$$

Boundary conditions and velocity-impulse assumption give:

$$\sqrt{\sigma_n} = n\pi; \quad \alpha_n = \frac{g_0^2 p_0 n^2 \pi^2}{12\nu w^2 \alpha}; \quad n = 1,3,5,\ldots$$

$$A_n = -\frac{x_0}{g_0}\sum_{\text{odd}\,n}\frac{4}{n\pi}\sin(n\pi\xi)\,e^{-\alpha_n t}$$

Force on the plate = $\displaystyle f_{sq}(t) = p_0 w l \int_0^1 \hat{p}(t,\xi)\,d\xi = -p_0 w l\,\frac{x_0}{g_0}\sum_{\text{odd}\,n}\frac{8}{\mathrm{n}^2\pi^2}e^{-\alpha_n t}$

Take the Laplace transform (continued on the next slide).

Finally, getting to lumped approximation…

$$F_{sq}(s) = \frac{96\nu\, l\, w^3}{\pi^4 g_0^3}\left\{\sum_{\text{odd } n}\left(\frac{1}{\mathrm{n}^4}\frac{1}{1+\dfrac{s}{\alpha_n}}\right)\right\} x_0 = \frac{96\nu\, l\, w^3}{\pi^4 g_0^3}\left\{\sum_{\text{odd } n}\left(\frac{1}{\mathrm{n}^4}\frac{1}{1+\dfrac{s}{\alpha_n}}\right)\right\} sX(s)$$

$$F_{sq}(s) = \frac{96\nu\, l\, w^3}{\pi^4 g_0^3}\frac{1}{1+\dfrac{s}{\omega_c}}\, sX(s) = \boxed{\frac{b}{1+\dfrac{s}{\omega_c}}}\, sX(s)$$

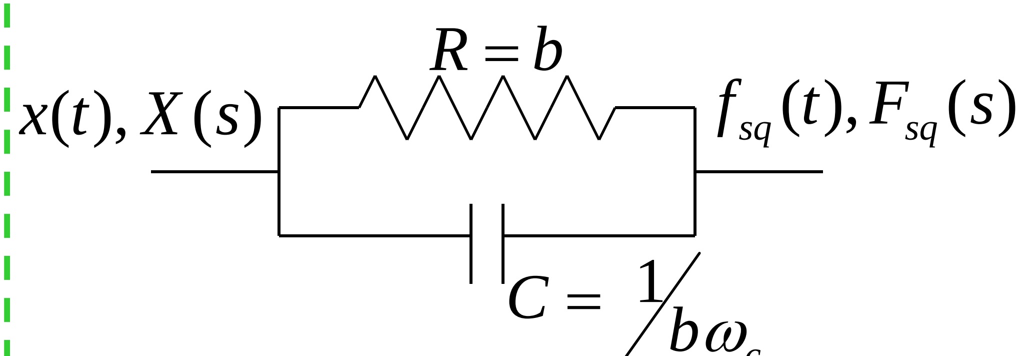
For $n = 1$ only.

Transfer function for general displacement input!

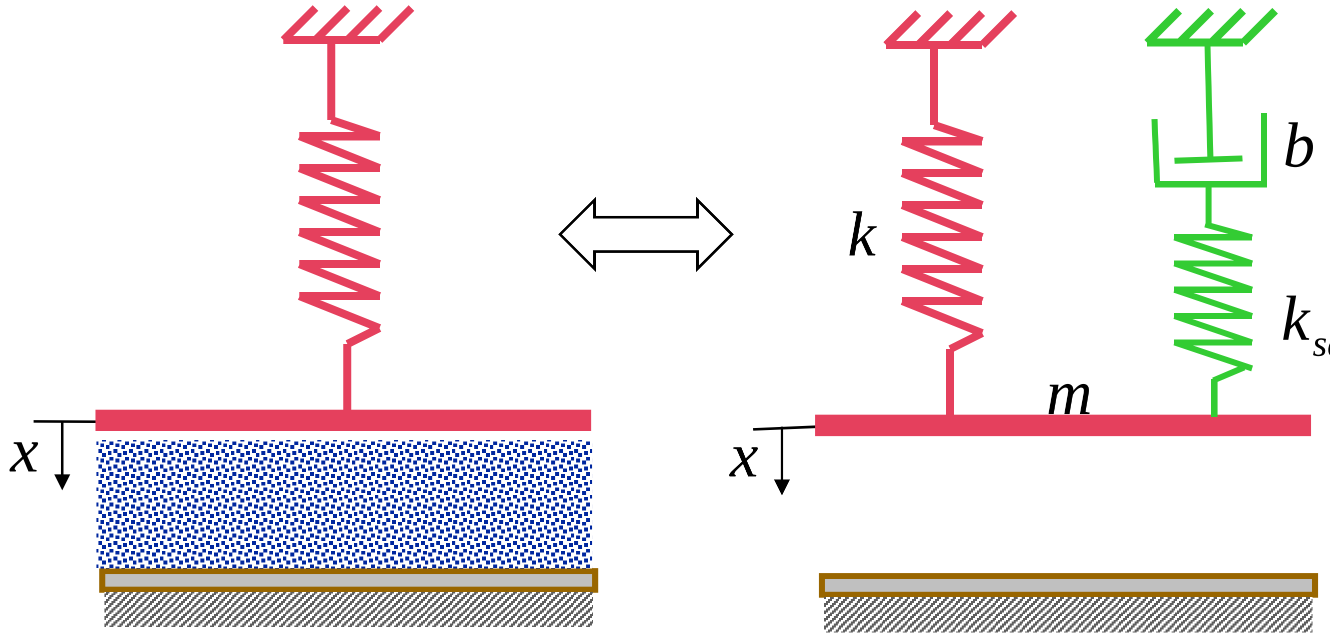
$$b = \frac{96\nu\, l\, w^3}{\pi^4 g_0^3} \qquad \omega_c = \frac{\pi^2 g_0^2 p_0}{12\nu\, w^2}$$

Damping coefficient

Cut-off frequency



What does it mean mechanically?



$$b = \frac{96 \nu\, l\, w^3}{\pi^4 g_0^3}$$

$$k_{sq} = b\omega_c = \frac{8 w p_0}{\pi^2 g_0}$$

Thus, squeezed film effect creates two effects:
Viscous damping + "air-spring"

Further analysis indicates that at low frequencies, damping dominates, and air-spring at high frequencies.

See S. D. Senturia, Microsystems Design, Kluwer, 2001, for details.

Move up to beam modeling…

$$\frac{\partial\left(p(x,y,t)\{g_0-u(x,t)\}\right)}{\partial t}=\frac{1}{12\nu}\nabla\cdot\left(p(x,y,t)\{g_0-u(x,t)\}^3\nabla p(x,y,t)\right)$$

$$\rho wt\frac{\partial u(x,t)}{\partial t^2}+\int_{-w/2}^{w/2}p(x,y,t)\,dy+EI\frac{d^4u(x,t)}{dx^4}-\underbrace{\frac{\varepsilon_0 wV^2(t)}{2\{g_0-u(x,t)\}^2}}=0$$

Solve these two coupled equations.

Note that this is still a parallel-plate approxmation!

An approach

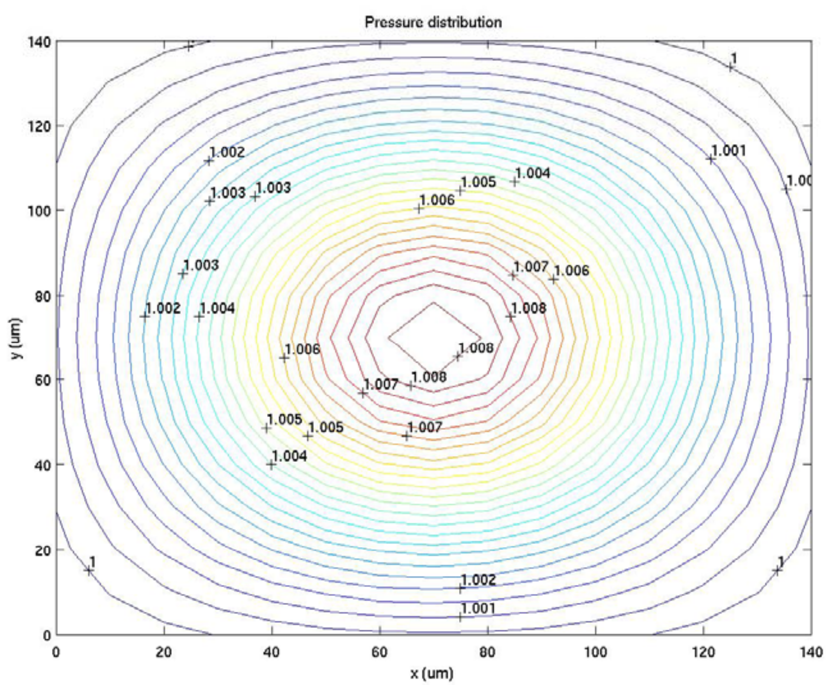
Use FDM for pressure equation and FEM or FDM for discretizing the dynamic equation, and integrate in time using the Runge-Kutta method.

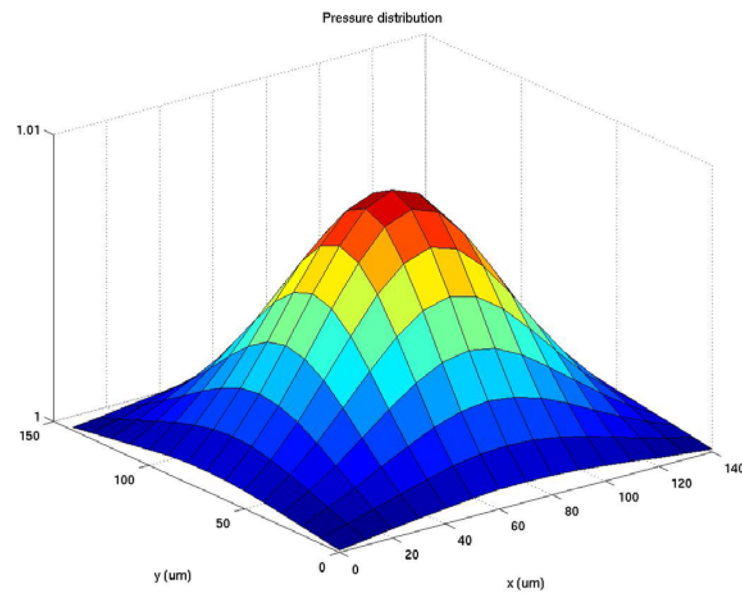
FD solution of Reynolds equation

$$12\eta \frac{\partial (Pg)}{\partial t} = \nabla.(g^3 P \nabla P)$$

Solution submitted by N. Sajinu for ME 237 course at IISc.

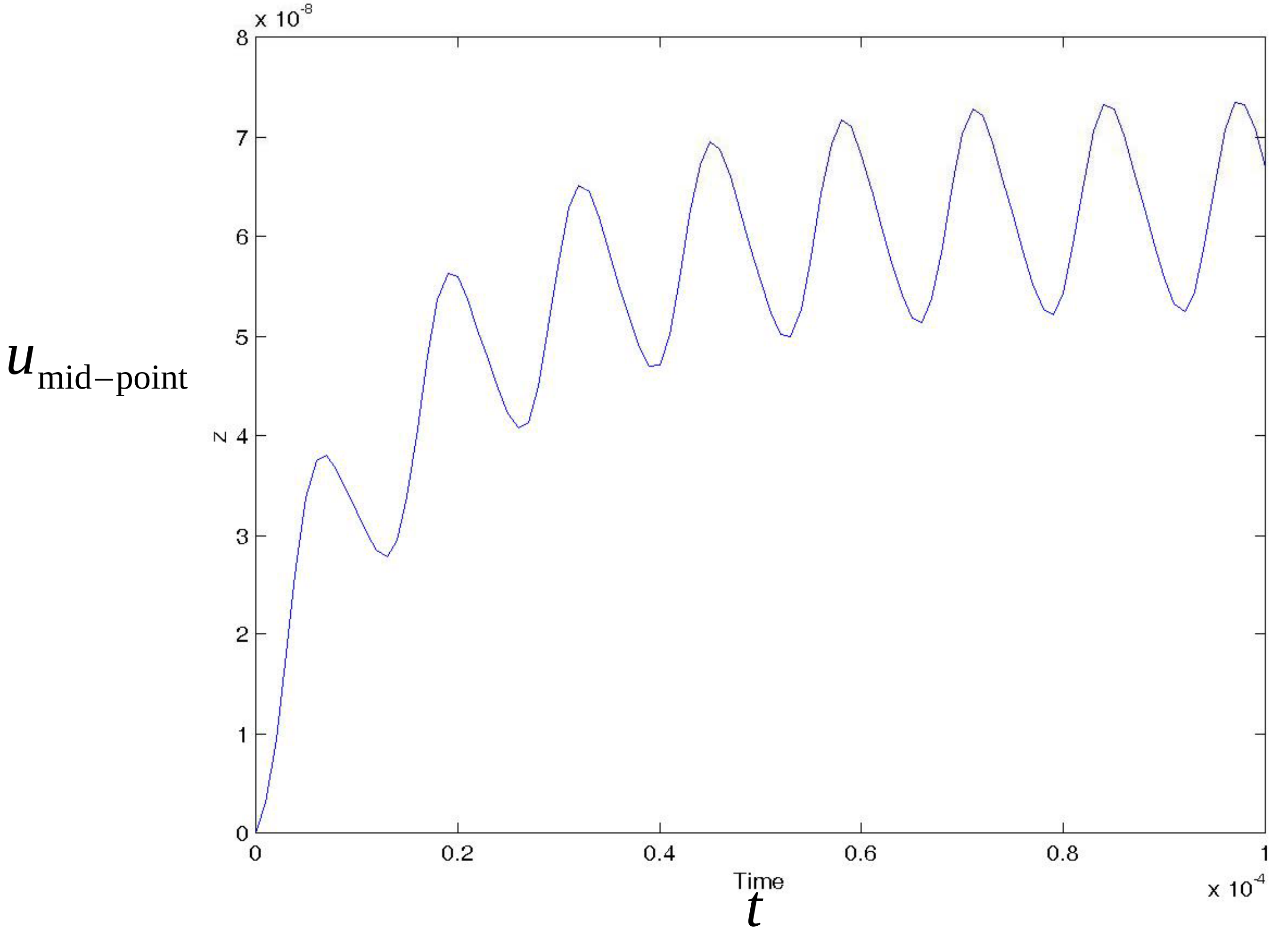
$$\frac{\partial P}{\partial t} = \frac{g^3}{12\eta}\left[\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2\right] + \frac{Pg^2}{12\eta}\left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial x^2}\right] + \frac{3Pg}{12\eta}\left[\frac{\partial P}{\partial x}.\frac{\partial g}{\partial x} + \frac{\partial P}{\partial y}.\frac{\partial g}{\partial y}\right] - \frac{P}{g}\frac{\partial P}{\partial t}$$



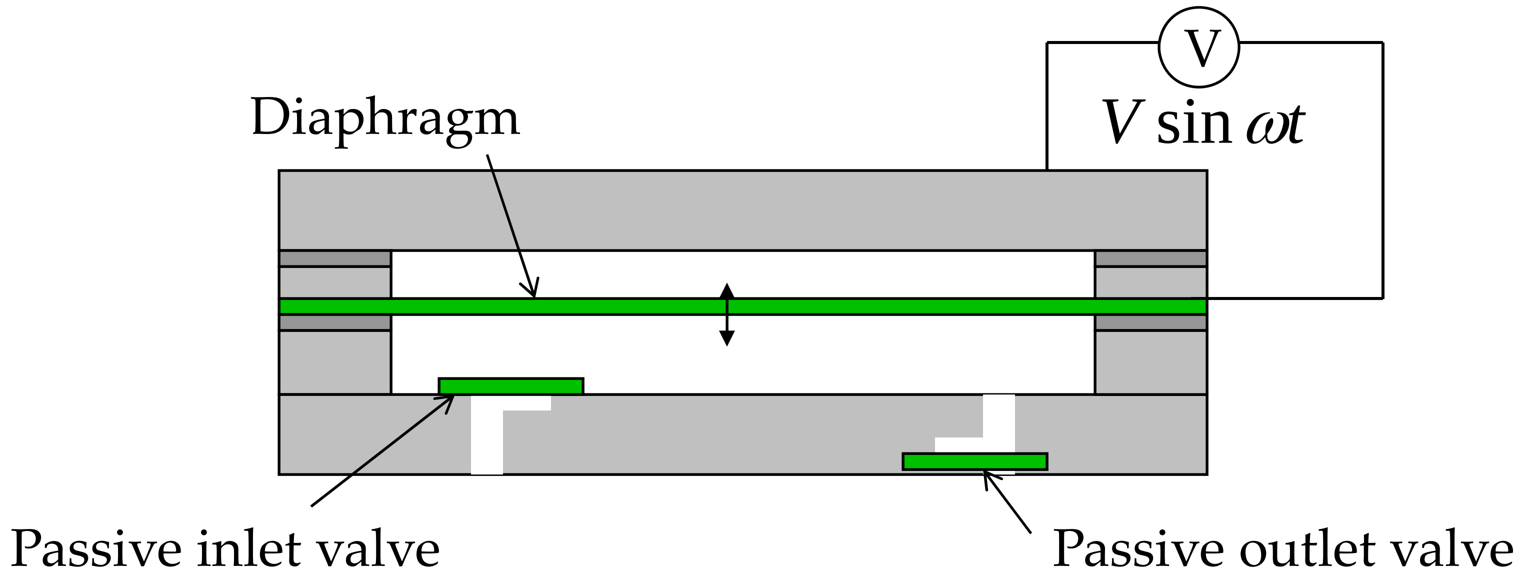


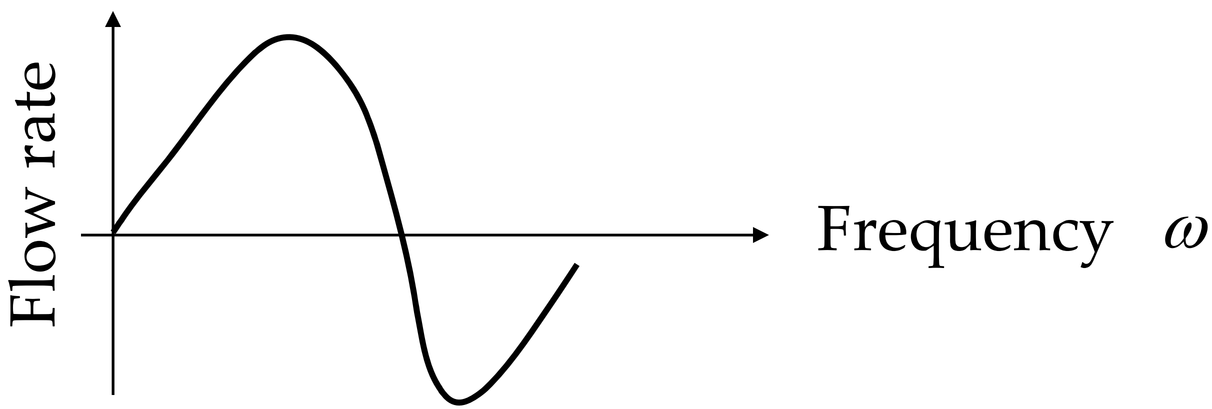
A typical beam's response with squeezed film effect

The transverse deflection of the mid-point of a fixed-fixed beam under (Vdc+Vac) voltage input under the squeezed film effect:



What about this problem now?





A problem involving three energy domains that are strongly coupled. Furthermore, the fluids part is non-trivial.

Main points

- MEMS are systems that tightly integrate many energetic phenomena, which makes their modeling non-trivial.
- Coupled multi-physics equations need to be solved.
- Reduced order lumped "macro" models are useful for design and system-level simulation