Step 1b (elastostatics) Compute the deformation of conductor 1 due to the electrostatic force. For this solve Eq. (6.33a) with the boundary equations taken from Eq. (6.34) to get the deformed geometry of conductor 1 as shown in Figure 6.11(c).

Step 2a (electrostatics) Subtract the deformed conductor 1 and conductor 2 and solve the electrostatic problem in the shaded region in Figure 6.11(d). Compute the changed forces on conductor 1 due to the newly computed electric potential, electric field, and surface charge.

Step 2b (elastostatics) Compute the deformed geometry of the conductor 1 due to the new forces applied on the undeformed configuration to get the new deformed geometry as shown in Figure 6.11(e).

Repeat the electrostatics and elastostatics steps until there is no more deformation of conductor 1. At this point, the electric potential will also remain unchanged. That is, we have obtained a self-consistent solution between the electrostatic and elastostatic equations.

▶ *FINITE ELEMENT FORMULATION FOR THE ELECTROSTATICS PROBLEM*

The general methodology of finite element analysis was described in Chapter 5. We now apply it to solve the partial differential equation (PDE) governing electrostatics [Eq. (6.23)], for a 3D domain of sufficiently large bounding box containing the conductors and dielectrics. Note that the discussion so far has focused on 2D domains only; the discussion that follows is more general and deals with 3D.

As is customary in FEA, we begin by writing the electrostatics PDE in the weak form by multiplying it with a trial function corresponding to the function to be solved and then integrating it over the domain. The function to be solved is $\phi(x, y, z)$. Let the weight function be $\phi_v(x, y, z)$. Then, we can write the weak form of the PDE in Eq. (6.23) as

$$\int_V \left(\varepsilon \nabla^2 \phi\right) \phi_v \ \mathrm{d}V = 0 \tag{6.35}$$

By using the fact that $(\varepsilon\nabla^2\phi)\phi_v = \varepsilon\nabla\cdot(\phi_v\nabla\phi) - \varepsilon(\nabla\phi\cdot\nabla\phi_v)$, Eq. (6.35) can be rewritten

$$\int_V \varepsilon\nabla\cdot(\phi_v\nabla\phi)\,\mathrm{d}V - \int_V \varepsilon(\nabla\phi\cdot\nabla\phi_v)\,\mathrm{d}V = 0 \tag{6.36}$$

By applying the divergence theorem [see Eq. (6.18)] to the first integral of Eq. (6.36), we can rewrite Eq. (6.36) as

$$\oint_S \varepsilon(\phi_v\nabla\phi\cdot\hat{\mathbf{n}})\mathrm{d}S - \int_V \varepsilon(\nabla\phi\cdot\nabla\phi_v)\,\mathrm{d}V = 0 \tag{6.37}$$

where $\hat{\mathbf{n}}$ is the outward normal to the closed surface S enclosing volume V. From Eqs. (6.28) and (6.40), we note that $\varepsilon\nabla\phi\cdot\hat{\mathbf{n}} = \varepsilon E_n = \psi_s$. Hence, $(\varepsilon\nabla\phi\cdot\hat{\mathbf{n}})$ in the first term of Eq. (6.37) can be replaced by ψ_s, the surface charge density. Thus, Eq. (6.37) assumes the form

$$\int_V \varepsilon(\nabla\phi\cdot\nabla\phi_v)\,\mathrm{d}V = \oint_S \psi_s \ \phi_v \ \mathrm{d}S \tag{6.38}$$

We now turn to interpolation of $\phi(x, y, z)$ and $\phi_v(x, y, z)$ using shape functions and nodal values of these quantities for a finite element (see Section 5.4.1). That is, within a finite element with p nodes, we have

$$\phi_e(x, y, z) = \sum_{i=1}^{p} N_i\phi_i = \mathbf{N}\boldsymbol{\varphi}_e \tag{6.39a}$$

$$\phi_{ve}(x, y, z) = \sum_{i=1}^{p} N_i\phi_i = \mathbf{N}\boldsymbol{\varphi}_{ve} \tag{6.39b}$$

where $\mathbf{N}$ is a $1 \times p$ shape function matrix and $\boldsymbol{\varphi}_e$ is the $p \times 1$ nodal potential vector. Note that the weight function ϕ_{ve} over the element is also interpolated using the same functions and the nodal vectors, $\boldsymbol{\varphi}_{ve}$, as in the Galerkin method described in Section 5.2.3.3.

We also write the gradients of the interpolated functions $\phi_e(x, y, z)$ and $\phi_{ve}(x, y, z)$ in Eqs. (6.39a) and (6.39b):

$$\begin{aligned} \nabla\phi_e &= \left(\frac{\partial\mathbf{N}}{\partial x}\hat{i} + \frac{\partial\mathbf{N}}{\partial y}\hat{j} + \frac{\partial\mathbf{N}}{\partial z}\hat{k}\right)\boldsymbol{\varphi}_e \text{ and} \\ \nabla\phi_{ve} &= \left(\frac{\partial\mathbf{N}}{\partial x}\hat{i} + \frac{\partial\mathbf{N}}{\partial y}\hat{j} + \frac{\partial\mathbf{N}}{\partial z}\hat{k}\right)\boldsymbol{\varphi}_{ve} \end{aligned} \tag{6.40}$$

By using Eqs. (6.39a, 6.39b,) and (6.40), we can write Eq. (6.38) for a finite element of volume V_e enclosed by the surface S_e as

$$\int_{V_e} \varepsilon(\nabla\phi_e \cdot \nabla\phi_e)\, \mathrm{d}V_e = \oint_{S_e} \psi_{se}\ \phi_{ve}\ \mathrm{d}S_e \tag{6.41a}$$

$$\int_{V_e} \varepsilon(\nabla\phi_e)^T(\nabla\phi_e)\, \mathrm{d}V_e = \oint_{S_e} \psi_{se}\ \phi_{ve}\ \mathrm{d}S_e \tag{6.41b}$$

where we have used the fact that the dot product of two vectors can be computed using the transpose operation on one of them. Thus, Eq. (6.417b) takes the form

$$\begin{aligned}\boldsymbol{\varphi}_e^T \Bigg[\int_{V_e} \varepsilon \Bigg\{ &\left(\frac{\partial \mathbf{N}}{\partial x}\right)^T \left(\frac{\partial \mathbf{N}}{\partial x}\right) + \left(\frac{\partial \mathbf{N}}{\partial y}\right)^T \left(\frac{\partial \mathbf{N}}{\partial y}\right) \\ &+ \left(\frac{\partial \mathbf{N}}{\partial z}\right)^T \left(\frac{\partial \mathbf{N}}{\partial z}\right) \Bigg\}\, \mathrm{d}V_e \Bigg] \boldsymbol{\varphi}_{ve} \\ &= \left\{ \oint_{S_e} \psi_{se}\ \mathbf{N}\ \mathrm{d}S_e \right\} \boldsymbol{\varphi}_{ve}\end{aligned} \tag{6.42}$$

Since $\boldsymbol{\varphi}_{ve}$ is arbitrary, Eq. (6.42) gives

$$\begin{aligned}\boldsymbol{\varphi}_e^T \Bigg[\int_{V_e} \varepsilon \Bigg\{ &\left(\frac{\partial \mathbf{N}}{\partial x}\right)^T \left(\frac{\partial \mathbf{N}}{\partial x}\right) + \left(\frac{\partial \mathbf{N}}{\partial y}\right)^T \left(\frac{\partial \mathbf{N}}{\partial y}\right) \\ &+ \left(\frac{\partial \mathbf{N}}{\partial z}\right)^T \left(\frac{\partial \mathbf{N}}{\partial z}\right) \Bigg\}\, dV_e \Bigg] \\ &= \left\{ \oint_{S_e} \psi_{se}\ \mathbf{N}\ \mathrm{d}S_e \right\}\end{aligned} \tag{6.43}$$

By taking the transpose, Eq. (6.43) can be rewritten as

$$C_e \boldsymbol{\varphi}_e = \mathbf{q}_e \tag{6.44a}$$

where

$$\begin{aligned}C_e = \Bigg[\int_{V_e} \varepsilon \Bigg\{ &\left(\frac{\partial \mathbf{N}}{\partial x}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{N}}{\partial x}\right) + \left(\frac{\partial \mathbf{N}}{\partial y}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{N}}{\partial y}\right) \\ &+ \left(\frac{\partial \mathbf{N}}{\partial z}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{N}}{\partial z}\right) \Bigg\}\, \mathrm{d}V_e \Bigg]\end{aligned} \tag{6.44b}$$

$$\mathbf{q}_e = \left\{ \oint_{S_e} \psi_{se}\ \mathbf{N}^{\mathrm{T}}\ \mathrm{d}S_e \right\} \tag{6.44c}$$

Note that $\mathbf{C}_e$ is the element capacitance matrix, $\boldsymbol{\varphi}_e$ the element potential vector, and $\mathbf{q}_e$ the element charge vector.

The next step is to assemble the element-level equations of Eqs. (6.44a)–(6.44c) into a global equation involving all the finite elements in a manner similar to the description in the last paragraph of Section 5.4.2. This yields

$$\mathbf{C}\boldsymbol{\varphi} = \mathbf{q} \tag{6.45}$$

Application of the boundary conditions becomes relevant at this stage. At any point in the domain, we can specify either the potential or the charge but not both. This is similar to specifying either force or displacement at any point in elastic analysis. By referring to Figures. 6.12(a) and (b) of a typical electrostatic problem and its coarse mesh for the purpose of illustration, we note that potential is known at the nodes 10, 11, 18, 19, 26, 27, 14, 15, 22, 23, 30, and 31, and that the specified charge is zero at all other nodes.

We then eliminate the ϕ's at the nodes where it is known, by deleting the corresponding rows and columns in Eq. (6.45), thus reducing it to a smaller size involving only the nodes where ϕ is not known. The resulting linear system is then solved. After that, the charges at the eliminated nodes are computed using Eq. (6.45).

To complete this derivation, it is pertinent to give the shape functions for the 2D problem so that the readers can implement the procedure to write their own computer program to solve the electrostatics problem in 2D. Shape functions for a three-noded [i.e. $p = 3$ in Eq. (6.39a)] triangular finite element are:

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y) \tag{6.46}$$

where

$$\begin{aligned}&a_1 = x_2y_3 - x_3y_2,\ b_1 = y_2 - y_3,\ c_1 = x_3 - x_2 \\ &a_2 = x_3y_1 - x_1y_3,\ b_2 = y_3 - y_1,\ c_2 = x_1 - x_3 \\ &a_3 = x_1y_2 - x_2y_1,\ b_3 = y_1 - y_2,\ c_3 = x_2 - x_1\end{aligned} \tag{6.47a}$$

and

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{6.47b}$$

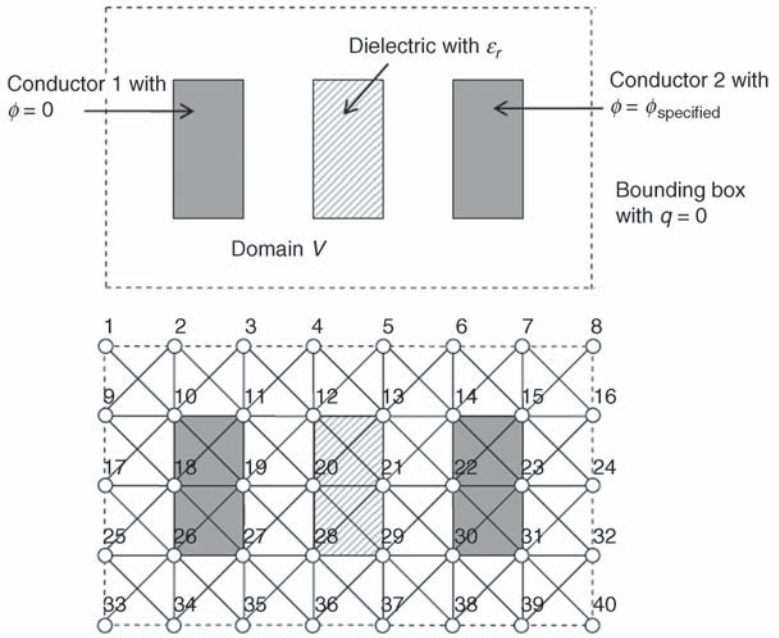

Figure 6.12 A sample problem in electrostatics. (a) The specifications, (b) the meshed model.

and (x_i, y_i, z_i) $(i = 1, 2, 3)$ are the coordinates of the triangular element. Using Eqs. (6.46) and (6.47), the element capacitance matrix of Eq. (6.44b) may be computed for the 2D triangular element of thickness t (see Figure 6.nnb) as

$$\mathbf{C}_e = \frac{\varepsilon t}{4A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1b_2 + c_1c_2) & (b_1b_3 + c_1c_3) \\ (b_1b_2 + c_1c_2) & (b_2^2 + c_2^2) & (b_2b_3 + c_2c_3) \\ (b_1b_3 + c_1c_3) & (b_2b_3 + c_2c_3) & (b_3^2 + c_3^2) \end{bmatrix} \tag{6.48}$$

The explicitly integrated element matrix of the preceding equation makes implementation of the FE program easy without resorting to numerical quadrature, as discussed in Section 5.4.7.

6.2.1 An Alternative Method for Solving the Coupled Problem

In Figure 6.11 and the discussion so far, we referred to the use of the FE or finite difference method to solve the electrostatic problem in a large bounding box from which the conductors are removed. For this, we used the differential form of Gauss's law [Eq. (6.19b) or its simplified form in Eq. (6.23) when the region does not enclose any charge]. Alternatively, we can also solve the integral form of Gauss's law given in Eq. (6.17).

Integral equations can be solved using the boundary element method just as differential equations are solved using the FE method. The advantage of using the boundary element method is that only the boundaries, and not the interiors of the objects need to be discretized with a mesh. Thus, in the problem of RF switch [Figs. 6.10(a) and (b)], we